Convection in a Porous Medium

ECMM713: Modelling Applications and Case Studies

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1 Introduction

Porous media allow for the movements of fluid through them. If there is a heat source such as a magma chamber at a depth within the medium this will then heat the fluid which, if the temperature difference ΔT between the heat source and the cooler upper boundary is large enough, may cause convection to take place. This occurs because the heated fluid starts to rise and must then be replaced by cooler fluid, which then in turn gets heated causing convection to commence.

In mid ocean ridges where the earths plates meet the convection of fluid accounts for the transfer of a large part of the Earth's heat flux. The water gets heated by the magma chambers which form at these ridges and carries the heat upwards into the surrounding ocean. This process forms hydrothermal vents known as Black Smokers and can form deposits of sulfide ore which the fluid picks up on its way through the rock and discharges as it cools when it comes out of the vent. Understanding this process of convection in systems such as this allows us to understand the process of heat transfer and the formation of these deposits.

In this case study we will look at how the governing equations for convection of a fluid in a two-dimensional convection cell are derived. We will also look at the stability of the steady state no-flow solution to the governing equations, and under what conditions this solution is stable and unstable in terms of the Rayleigh number. Towards the end we will look at the model in the context of Black Smokers and look at why there appears to be an upper limit for the fluid venting from these systems.

2 Governing Equations

2.1 Darcy's Law

In a porous medium, there are two ways to define the rate of flow. There is the surface velocity $\vec{u}$ which is the rate at which a free surface of fluid above the porous medium would move and the interstitial velocity $\vec{v}$ which is the velocity of the water between the particles of the rock (see Figure 1). The interstitial velocity has the form $\vec{v} = \frac{\vec{u}}{\phi}$ where $\phi \in [0, 1]$ is the volume fraction of the interstices.

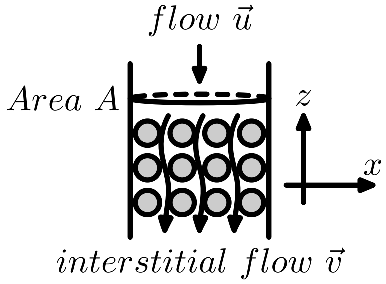

Figure 1: The two rates of flow; the total flow $\vec{u}$ and the interstitial flow $\vec{v}$

If we now look at the flow between the interstices, we can look at it as a 2-dimensional flow between two plates flowing in the x direction driven by a pressure gradient $\frac{\partial p}{\partial x} = p_x$ in the x direction. The flow will only have a component in the x direction which will be a function of z (see Figure 2).

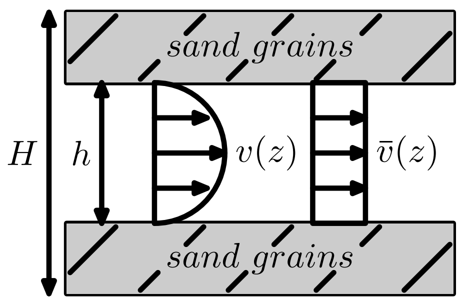

Figure 2: Flow between the interstices of the sand grains as 2-d flow between plates where h is the average distance between interstices, H is the interstitial width plus the width of a sand grain, $v(z)$ is the fluid flow between the interstices and $\bar{v}(z)$ is the average flow. In this case $\phi = \frac{h}{H}$.

The Navier Stokes equations for a fluid with velocity $\vec{v}$ and pressure p in two dimensions are

$$\vec{v} = \begin{pmatrix} v(z) \\ 0 \end{pmatrix} \text{ satisfying } \nabla \cdot \vec{v} = 0 \qquad \text{(mass conservation)} \tag{2.1}$$

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} \right) = -\nabla p + \mu \nabla^2 \vec{v} \qquad \text{(momentum conservation)} \tag{2.2}$$

where μ is the dynamic viscosity of the fluid measured in $[Pa \cdot s]$ and ρ is the fluid density in $[kg \cdot m^{-3}]$. In this plane parallel flow we have no time dependence, and the only flow is in the x direction as a function of z only, so $\frac{\partial \vec{v}}{\partial t} = (\vec{v} \cdot \nabla) \vec{v} = 0$ and $\nabla^2 \vec{v}$ reduces to $\frac{\partial^2 v}{\partial z^2} \equiv v_{zz}$ meaning that mass conservation (equation (2.1)) is satisfied and equation (2.2) becomes

$$0 = -\nabla p + \mu v_{zz}. \tag{2.3}$$

We also know that the pressure gradient only exists in the x direction, meaning that $\nabla p = p_x$ giving us a second order differential equation for $v(z)$:

$$0 = -p_x + \mu v_{zz}. \tag{2.4}$$

We can then integrate this to solve for $v(z)$ using the non-slip boundary conditions $v(0) = v(h) = 0$, since we are considering a viscous fluid, giving us

$$v(z) = -\frac{p_x}{2\mu} z(h - z). \tag{2.5}$$

We can now define the average flow velocity inside the pore, $\bar{v}$, as

$$\bar{v} = \frac{1}{h}\int_0^h v(z)\,dz. \tag{2.6}$$

Putting $v(z)$ into this and integrating then gives us

$$\bar{v} = -\frac{h^2}{12\mu} p_x. \tag{2.7}$$

From considering the interstitial velocity we have a relationship between the average flow in the interstices and the surface velocity of the form $\vec{u} = \phi\vec{\bar{v}}$. Since the flow is one dimensional we can drop the vectors without loss of generality, giving us the relation $u = \phi\bar{v}$, which we can then substitute our value for $\bar{v}$ into to give us

$$u = -\left[\frac{\phi h^2}{12}\right]\frac{1}{\mu} p_x. \tag{2.8}$$

Here we can then define $\left[\frac{\phi h^2}{12}\right] = k$ where k is called the "permeability" of the rock which has units $[m^2]$. In general the permeability has the form $k = \frac{\delta\phi h^2}{12}$ where h is the typical width of the pores and $\delta = O(1)$ is a constant dependent on the geometry of the voids. For a plane parallel void, which is the case here, $\delta = 1$ and the permeability k takes the form $k = \frac{\phi h^2}{12}$.

2.1.1 Hydrostatic Pressure

We will now look at the hydrostatic pressure of a fluid in equilibrium. We take a small volume of fluid at rest, with height δz (see Figure 3).

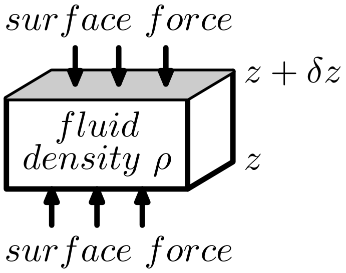

Figure 3: Small volume of fluid in vertical equilibrium.

The net surface force in the z-direction will then be

$$A\left[p(z) - p(z+\delta z)\right] \tag{2.9}$$

where A is the area of a horizontal cross section of the fluid. The net volume force in the z-direction will be

$$-\rho g A \delta z. \tag{2.10}$$

Since the fluid is at rest, these forces must be in balance, so we will equate these yielding

$$\frac{p(z+\delta z) - p(z)}{\delta z} = -\rho g. \tag{2.11}$$

We can then let $\delta z \to 0$ which will give us the hydrostatic balance equation

$$\frac{\partial p}{\partial z} = -\rho g \tag{2.12}$$

where g is acceleration due to gravity in $[m \cdot s^{-2}]$

2.1.2 Darcy's Law in 2 Dimensions

Looking at equation (2.8), we can generalise this to multi-dimensional flow:

$$\vec{u} = -\frac{k}{\mu}\nabla p. \tag{2.13}$$

From the discussion in Section 2.1.1 above we can see that in the vertical direction the net pressure will not be equal to the real pressure in the system. It will instead be equal to the net force of the real pressure and the gravitational pressure. We can then define the reduced pressure $P = p - \rho g z$ where p is the real pressure of the system, since we assume hydrostatic conditions, so from the previous section $p_z = -\rho g$. Putting reduced pressure in the place of pressure in equation (2.13) then gives us the 2-dimensional form of Darcy's Law

$$\vec{u} = -\frac{k}{\mu}\left(\nabla(p - \rho g z)\right) \tag{2.14}$$

$$\Rightarrow \vec{u} = -\frac{k}{\mu}\left(\nabla p - \rho \vec{g}\right). \tag{2.15}$$

Here the vector $\vec{g}$ is taken to be

$$\vec{g} = \begin{pmatrix} 0 \\ -g \end{pmatrix}. \tag{2.16}$$

Taking the individual components of Darcy's Law (equation (2.15)), with

$$\vec{u} = \begin{pmatrix} u \\ w \end{pmatrix} \tag{2.17}$$

gives us two equations for the velocity:

$$u = -\frac{k}{\mu} p_x \tag{2.18}$$

$$w = -\frac{k}{\mu}(p_z + \rho g). \tag{2.19}$$

These are the equations for the Darcy velocity in the presence of gravity and a pressure gradient.

2.1.3 Conservation Laws

In order to derive the governing equations for the system we will need to use the conservation of mass and energy equations. For a general quantity q, with flux of q equal to $\vec{F}$ having units of q per unit area per unit time the conservation law tells us

$$\frac{d}{dt}\int_V q\, dV = -\int_S \vec{F} \cdot d\vec{n}\, dS \tag{2.20}$$

where V is an arbitrary volume enclosed by the surface S with unit normal $\vec{n}$. This then can be written as

$$\int_V \frac{\partial q}{\partial t}\, dV = \int_V -\nabla \cdot \vec{F}\, dV \tag{2.21}$$

by the divergence theorem. Since the volume V is arbitrary, the integrands must be equal to one another leaving us with the equality

$$\frac{\partial q}{\partial t} = -\nabla \cdot \vec{F}. \tag{2.22}$$

To derive the conservation of mass equation we have $q = \rho$, meaning $\vec{F} = \rho\vec{u}$ giving the equation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho\vec{u}) = -\rho\nabla \cdot \vec{u} - \vec{u} \cdot \nabla\rho. \tag{2.23}$$

The Boussinesq approximation for the density of a fluid states that other than when it appears in a buoyancy term, which in this case is a term multiplied by the gravity term g, variations in the density may be neglected due to the fact that they are sufficiently small.[5] When the density is not neglected it can be approximated by a linear function of temperature. We will make use of this approximation in mass and energy conservation. Using the fact that ρ is constant in time ($\frac{\partial\rho}{\partial t} = 0$) equation (2.23) then reduces to

$$\nabla \cdot \vec{u} = 0 \tag{2.24}$$

which is the conservation of mass equation.

We can do the same for energy by setting $q = \rho c_p(T - T_0)$ where c_p is the specific heat capacity in $[J \cdot kg^{-1} \cdot K^{-1}]$ and T_0 is the reference temperature of the fluid and is a constant with units $[K]$. Here heat can transfer through conduction and convection, meaning that

$\vec{F} = \rho c_p(T - T_0)\vec{u} - \lambda \nabla T$ where $\rho c_p(T - T_0)\vec{u}$ is the convective heat flux, $\lambda \nabla T$ is the conductive heat flux, and λ is the thermal conductivity in $[W \cdot m^{-1} \cdot K^{-1}]$. This then gives us

$$\frac{\partial}{\partial t}\left(\rho c_p \left(T - T_0\right)\right) = -\nabla \cdot \left(\rho c_p \left(T - T_0\right)\vec{u} - \lambda \nabla T\right). \tag{2.25}$$

Again ρ is constant in time and space, and T_0 is constant, so this can be reduced to

$$\rho c_p \frac{\partial T}{\partial t} = -\rho c_p \left[\vec{u} \cdot \nabla T + \left(\nabla \cdot \vec{u}\right) T\right] + \lambda \nabla^2 T. \tag{2.26}$$

Now using the mass conservation equation (2.24) we can further simplify and rearrange to get

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T = \left[\frac{\lambda}{\rho c_p}\right] \nabla^2 T. \tag{2.27}$$

Here we set $\left[\frac{\lambda}{\rho c_p}\right] = D$ where D is called the thermal diffusivity with units $[m^2 \cdot s^{-1}]$.

2.2 The Governing Equations

For the density of the fluid, so far we have used the Boussinesq Approximation so have not had to worry about what form the density takes, however in equation (2.15) we have the density term multiplied by g, so we now need an expression for density. We will take the constitutive relation for the density to be a linear approximation of the form[2]

$$\rho = \rho_0 \left[1 - \alpha \left(T - T_0\right)\right] \tag{2.28}$$

where $\alpha \approx 10^{-4}\ K^{-1}$ is the thermal expansivity of the fluid and $\rho_0 \approx 1000\ kg \cdot m^{-3}$ is the reference density of the fluid.

Summarising our equations so far then, we have:

	$\rho = \rho_0 \left[1 - \alpha(T - T_0)\right]$	(Constitutive Relation)	(2.29)
ρ linear in T	$\vec{u} = -\frac{k}{\mu}\left(\nabla p - \rho \vec{g}\right)$	(Darcy's Equation)	(2.30)
$\rho \approx \rho_0$	$\nabla \cdot \vec{u} = 0$	(Mass Conservation)	(2.31)
$\rho \approx \rho_0$	$\frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T = D \nabla^2 T$	(Energy Conservation)	(2.32)

Substituting the linear approximation of ρ into Darcy's Equation then gives us

$$\vec{u} = \frac{k}{\mu}\left(\nabla p - \rho_0 \left[1 - \alpha(T - T_0)\right]\vec{g}\right). \tag{2.33}$$

We now have three equations in four unknowns; u, w, p and T. We can reduce this to two equations in two unknowns by defining a stream function as[6, pg.41]

$$\psi_z = u \tag{2.34}$$
$$-\psi_x = w. \tag{2.35}$$

This clearly satisfies mass conservation since

$$\nabla \cdot \vec{u} = \psi_{xz} - \psi_{zx} = 0. \tag{2.36}$$

We can then eliminate pressure from the equations for mathematical convenience by first substituting ψ into the separate components of Darcy's Law which becomes

$$\psi_z = -\frac{k}{\mu} p_x \tag{2.37}$$

$$-\psi_x = -\frac{k}{\mu}(p_z + \rho_0 \left[1 - \alpha(T - T_0)\right] g). \tag{2.38}$$

Cross differentiating these and taking their difference then eliminates pressure leaving us with the first of our governing equations

$$\nabla^2 \psi = \psi_{xx} + \psi_{zz} = \frac{kg}{\mu} \rho_0 T_x. \tag{2.39}$$

We must now substitute the stream function into the conservation of energy equation in order to reduce it to an equation with only two unknowns which will be the second of our governing equations:

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T = D \nabla^2 T \tag{2.40}$$

$$\Rightarrow \qquad \frac{\partial T}{\partial t} = \dot{T} = -\vec{u} \cdot \nabla T + D \nabla^2 T \tag{2.41}$$

$$\Rightarrow \qquad \dot{T} = -\psi_z T_x + \psi_x T_z + D \nabla^2 T. \tag{2.42}$$

3 Non-Dimensionalisation

We now have our governing equations and want to non-dimensionalise them. This will allow us to consolidate all of the physical constants in our equations into a single new constant. This will mean that when we look at numerical solutions for the system for a given value of this new variable, our analysis will apply to a whole range of physical systems whose physical constants are in the ratio of this new constant rather than just one unique system. It will also mean that we will not have to worry about what physical size any of our individual constants or variables will take for any given system to be feasible.

To do this we will define new dimensionless variables

$$\hat{z} = \frac{z}{L} \qquad \hat{x} = \frac{x}{L} \qquad \hat{T} = \frac{T - T_0}{\Delta T}$$

$$\hat{t} = \frac{D}{L^2} t \qquad \hat{\psi} = \frac{\mu}{kg\alpha\rho_0 L \Delta T} \psi \qquad \hat{\nabla} = L \nabla$$

where a $\hat{}$ represents a dimensionless variable, L is the height of the convection cell and ΔT is the drop in temperature between the top and bottom of the convection cell.

Substituting these into our governing equations then yields

$$
\begin{aligned}
&(2.39) \qquad && \nabla^2\psi = -\frac{kg\alpha\rho_0}{\mu}T_x \\
&\Rightarrow && \hat{\nabla}^2\hat{\psi} = -\hat{T}_{\hat{x}} && (3.1)
\end{aligned}
$$

and

$$
\begin{aligned}
&(2.42) \qquad && \frac{\partial T}{\partial t} = -\psi_z T_x + \psi_x T_z + D\left(T_{xx} + T_{zz}\right) \\
&\Rightarrow && \frac{\partial \hat{T}}{\partial \hat{t}} = \left[\frac{kg\alpha\rho_0 L\Delta T}{\mu D}\right]\left(-\hat{\psi}_{\hat{z}}\hat{T}_{\hat{x}} + \hat{\psi}_{\hat{x}}\hat{T}_{\hat{z}}\right) + \hat{\nabla}^2\hat{T} && (3.2)
\end{aligned}
$$

where we set $\left[\frac{kg\alpha\rho_0 L\Delta T}{\mu D}\right] = Ra$ where Ra is the Rayleigh number.
So the two non-dimensional governing equations are

$$
\begin{cases} \hat{\nabla}^2\hat{\psi} = -\hat{T}_{\hat{x}} \\ \hat{T}_{\hat{t}} = Ra \cdot \left(-\hat{\psi}_{\hat{z}}\hat{T}_{\hat{x}} + \hat{\psi}_{\hat{x}}\hat{T}_{\hat{z}}\right) + \hat{\nabla}^2\hat{T}. \end{cases} \tag{3.3}
$$

We now have our two equations to solve to find the structure of convection through the system. In order to be able to do this we will need to impose some boundary conditions.

4 Boundary Conditions

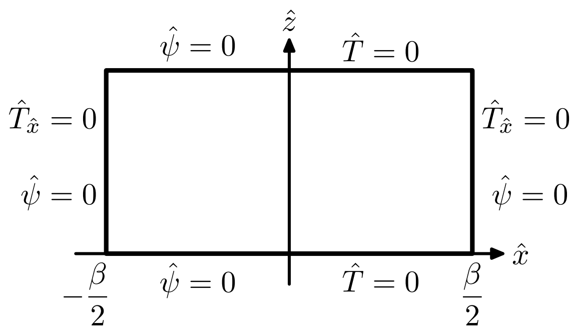

Figure 4: Convection cell with with impermeable boundaries, fixed temperature at the top and bottom and insulating side boundaries.

In order to be able to solve equations (3.3) numerically we need some boundary conditions for the system. We will be looking at a rectangular convection cell (See Figure 4), with the vertical coordinate $\hat{z}$ running between 0 and 1, corresponding to a cell with height $z = L$, and the width of the cell will be $L\beta$ meaning that the horizontal coordinate $\hat{x}$ will be between $-\frac{\beta}{2}$ and $\frac{\beta}{2}$. We will want the cell to have impermeable boundaries, so that the amount of fluid within the cell will remain constant. This then tells us that on the boundaries $\hat{x} = \pm\frac{\beta}{2}$ we must have $\hat{\psi}_{\hat{x}} = 0$ and on $\hat{z} = 0$ and $\hat{z} = 1$ we have $\hat{\psi}_{\hat{z}} = 0$. This then tells us that $\hat{\psi}$ is constant on the boundaries of the cell so without loss of generality we can require $\hat{\psi} = 0$ on all boundaries. We want the cell to have a negative temperature gradient in the vertical direction so that the bottom of the cell is hotter than the top by an amount of ΔT, giving us $\hat{T} = 1$ on $\hat{z} = 0$ and $\hat{T} = 0$ on $\hat{z} = 1$. Lastly we want the boundaries of the cell to be insulating, which corresponds to having no heat flux across any of the boundaries, so we have $\hat{T}_{\hat{x}} = 0$ on $\hat{x} = \pm\frac{\beta}{2}$.

5 Stability Analysis

There is a solution to the governing equations (3.3) satisfying the boundary conditions where no flow exists, and the transfer of heat is through conduction only;

$$\begin{cases} \hat{\psi} = 0 \\ \hat{T} = 1 - \hat{z}. \end{cases} \tag{5.1}$$

We want to know if this equilibrium state is stable or unstable, since if it is unstable then we can have convection, however in the case that it is stable all initial conditions will decay to this state for $t \to \infty$. We can do this by perturbing the solution slightly from the basic state and looking at how this perturbation behaves as t grows. We will look at a sinusoidal perturbation of the form

$$\begin{cases} \hat{\psi} = 0 + \tilde{\psi} e^{il\hat{x}} e^{n\hat{t}} f(\hat{z}) \\ \hat{T} = 1 - \hat{z} + \tilde{T} e^{il\hat{x}} e^{n\hat{t}} g(\hat{z}) \end{cases} \tag{5.2}$$

where $|\tilde{\psi}| \ll 1$, $|\tilde{T}| \ll 1$ and l is the horizontal wavenumber of the system. We now plug these perturbed solutions into the governing equations (3.3) giving us

$$\tilde{\psi}\left(-l^2 f(\hat{z}) + f''(\hat{z})\right) e^{il\hat{x}} e^{n\hat{t}} = -il\tilde{T} e^{il\hat{x}} e^{n\hat{t}} g(\hat{z}) \tag{5.3}$$

$$n\tilde{T} e^{il\hat{x}} e^{n\hat{t}} g(\hat{z}) = Ra \cdot \left[-il\tilde{\psi} e^{il\hat{x}} e^{n\hat{t}} f(\hat{z})\right] + \tilde{T}\left(-l^2 e^{il\hat{x}} e^{n\hat{t}} g(\hat{z}) + e^{il\hat{x}} e^{n\hat{t}} g''(\hat{z})\right), \tag{5.4}$$

which then reduces to

$$\begin{cases} \tilde{\psi}\left(-l^2 f(\hat{z}) + f''(\hat{z})\right) = -il\tilde{T} g(\hat{z}) \\ n\tilde{T} g(\hat{z}) = Ra \cdot \left[-il\tilde{\psi} f(\hat{z})\right] + \tilde{T}\left(-l^2 g(\hat{z}) + g''(\hat{z})\right). \end{cases} \tag{5.5}$$

We can try the solution $f(\hat{z}) = g(\hat{z}) = sin(s\hat{z})$ where we require the boundary condition $\hat{\psi} = \hat{T} = 0$ on $\hat{z} = 1$ to hold yielding $s = r\pi \;\; r \in \mathbb{N}$. Here s is the vertical wavenumber, meaning that $s = 2\pi/\text{wavelength}$, which tells us that r is the number of half wavelengths between the boundaries $\hat{z} = 0$ and $\hat{z} = 1$. We must also have $\hat{\psi} = \hat{T}_{\hat{x}} = 0$ on the horizontal boundaries $\hat{x} = \pm\frac{\beta}{2}$ which means that we must have an integer number of half wavelengths; $\beta = q * \frac{1}{2}\text{wavelengths} = q\pi/l$, $q \in \mathbb{N}$. This then gives us $l = q\pi/\beta$. We can then see that as we take β larger we have more possibilities for the value of l, and in the limit $\beta \to \infty$ it may take any real value.

We can now substitute $f(\hat{z}) = g(\hat{z}) = sin(s\hat{z})$ into (5.5) giving us

$$\tilde{\psi}\left(l^2 + s^2\right) = il\tilde{T} \tag{5.6}$$

$$n\tilde{T} = Ra \cdot \left[-il\tilde{\psi}\right] + \tilde{T}\left(-l^2 + s^2\right). \tag{5.7}$$

We can change this into the matrix equation

$$\begin{pmatrix} il & -\left(l^2+s^2\right) \\ n+\left(l^2+s^2\right) & ilRa \end{pmatrix} \begin{pmatrix} \tilde{T} \\ \tilde{\psi} \end{pmatrix} = \vec{0} \tag{5.8}$$

which tells us that the determinant of the matrix, $-l^2Ra + (l^2+s^2)\left[n+(l^2+s^2)\right]$, is equal to 0 since $\tilde{\psi} \neq 0 \neq \tilde{T}$ so the matrix is not invertible. This then gives us

$$-l^2Ra + \left(l^2+s^2\right)\left[n+\left(l^2+s^2\right)\right] = 0 \tag{5.9}$$

$$\Rightarrow \qquad n = \frac{l^2}{l^2+s^2}Ra - \left(l^2+s^2\right) \tag{5.10}$$

and setting $\sigma = l^2 + s^2$ yields

$$n = \frac{(\sigma - s^2)Ra - \sigma^2}{\sigma}. \tag{5.11}$$

We now have a relation between n and the horizontal and vertical wavenumbers, so if we plot n against σ we will have a series of curves of constant s (see Figure 5).

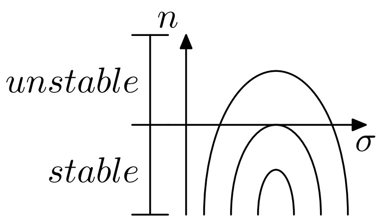

Figure 5: Curves of constant s plotted against n, with stable and unstable regions indicated.

For values of $n < 0$ we will have a stable solution with all perturbations decaying to 0 as $t \to \infty$ and for $n > 0$ we will have an unstable solution with perturbations growing as t gets larger. The case of marginal stability will then be when $n = 0$, this curve of constant s will have only one point touching the σ axis, so at this point $\frac{dn}{ds} = 0$. Looking at this curve we get 2 equations;

$$0 = \frac{(\sigma - s^2)Ra - \sigma^2}{\sigma} \tag{5.12}$$

$$0 = \frac{\sigma\left(Ra - 2\sigma\right) - \left(\sigma - s^2\right)Ra - \sigma^2}{\sigma^2} \tag{5.13}$$

Substituting equation (5.12) into equation (5.13) we then have the two relationships

$$Ra = 2\sigma \tag{5.14}$$

$$\left(\sigma - s^2\right)Ra = \sigma^2. \tag{5.15}$$

Substituting the first of these into the second and simplifying then gives us that $\sigma = 2s^2$, which in turn gives us

$$Ra = 4s^2. \tag{5.16}$$

Our restriction on the value of s then tells us that $Ra = 4r^2\pi^2$, which then tells us that the smallest possible Rayleigh number at marginal stability is

$$Ra = 4\pi^2. \tag{5.17}$$

So we now have a condition on the Rayleigh number as to whether the steady state, zero flow solution to our governing equations is stable for all wavenumbers, or if there is some combination of wavenumbers for which it is unstable; if $Ra < 4\pi$ then the solution is stable for all wavenumbers l and s, but if $Ra > 4\pi$ then the solution will be unstable for at least one combination of l and s depending on the exact value of the Rayleigh number Ra.

6 Extension

The simple model for convection in a porous medium discussed above can be adapted to model the behavior of the convecting fluid in sea floor hydrothermal vents such as Black Smokers. More complex models for such convection systems are discussed in two-dimensions in [3] and [4] and in three-dimensions in [1]. The first modification required to adapt the model is that for the case of black smokers the top boundary of the convection cell at $\hat{z} = 1$ will no longer be impermeable. Instead there must be a flow of fluid through this boundary corresponding to the hot fluid venting upward through the sea floor and being replaced by cold seawater flowing downward into the rock. The second modification is that the shape of the heating on the lower boundary $\hat{z} = 0$ can be changed from being uniform across the whole cell to a localised heating having a Gaussian profile. This localised heating corresponds to a localised magma chamber which is the heat source driving the convection in these systems. This second adaptation isn't strictly necessary; if the adaptation is made then simulations will model the across-axial region of the system, whereas if it is not made then the model will be of convection cells along the axis.

In sea floor hydrothermal vents there is an upper limit on the temperature of the fluid flowing from the vent; indeed, no vent above a temperature of $405^\circ\ C$ has been observed in a physical system.[4] Using the Boussinesq approximation for the density of the fluid does not impose this upper limit on the temperature since the hydrothermal properties of the fluid at these temperatures are not taken into account in this approximation, but are believed to be the reason that this upper limit exists.

In a three dimensional convection system the expected structure of the convecting fluid can be viewed as three separate regions (see Figure 6). There is a downward flowing region with a cross-sectional area of A_d, an upward flowing region having a cross-sectional area A_u and a connecting region between these where the heating of the fluid takes place which is sometimes termed the reaction zone (c.f. Ref. [4]). All fluid which enters the cell must eventually leave it, therefore we must have mass balance between the up- and down-flowing fluid:

$$A_u \rho_u \frac{k_u}{\mu_u}(p_z - \rho_u g) = -A_d \rho_d \frac{k_d}{\mu_d}(p_z - \rho_d g) \tag{6.1}$$

where a subscript u indicates properties of up-flowing fluid and d the down-flow. This will only be true if the pressure gradient in the up- and down-flowing regions is zero in the horizontal direction. This will be true since there is very little horizontal flow of fluid in these regions.[1]

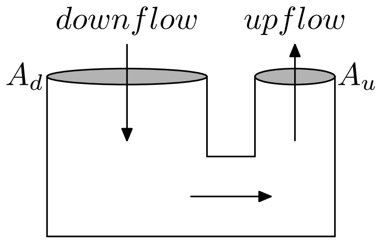

Figure 6: Structure of convection cell

If we then write $\frac{\partial p}{\partial z} = g\rho_n$ following *Coumou et al.* [1], where ρ_n is the density of a fictitious neutrally buoyant fluid then the equation (6.1) can be used to derive an expression for how well the fluid can transport heat, termed the *'fluxibility'* of the fluid:

$$F = \frac{\rho_u(h_u - h_d)(\rho_d - \rho_u)}{\mu_u}\frac{1}{1+\gamma R} \tag{6.2}$$

where $\gamma = \dfrac{A_u k_u}{A_d k_d}$, $R = \dfrac{\mu_d \rho_u}{\mu_u \rho_d}$ and h is the thermal enthalpy of the fluid. In two dimensional flow the size of the cross sectional areas A_u and $A_d \to 0$, meaning that the definition of the fluxibility can be reduced to

$$F = \frac{\rho_u(h_u - h_d)(\rho_d - \rho_u)}{\mu_u} \tag{6.3}$$

which is the form derived by Jupp and Schultz in [4]. The value of this variable depends on both the density and dynamic viscosity of the fluid as well as its specific enthalpy h, all of which are functions of temperature. To understand how the fluxibility varies with temperature we must look at how each of its component parts varies with temperature. The first of these is density, which in our simple model we have approximated using the Boussinesq approximation.

In the analysis of the different component parts of the fluxibility, the properties of pure water are usually examined rather than that of a brine which would be found in the physical system (for example in [1], [3], [4]). We do this because the properties of the two fluids are similar although using pure water allows for a simplification of the problem. This is due to the fact that for pressures higher than the critical pressure of 22 MPa pure water cannot boil as it would for lower pressures. Instead it changes from a liquid like state to a gas like state in a smooth manner. The pressures found at the sea floor will typically exceed this critical pressure meaning that the water in this system can only exist in a single physical state, allowing us to ignore the added complication that water appearing in multiple physical states at the same time would induce.

To see how the density of the fluid changes with increasing temperature we must look at the validity of the Boussinesq approximation for the density of the fluid in the conditions of hydrothermal sea floor convection systems. The Boussinesq approximation assumes that wherever variation of density cannot be neglected that it takes the form of a linear function of temperature. This approximation is reasonable for fluid at relatively low temperatures ($T < 200° \ C$), although for higher temperatures we run into problems if we try to incorporate it into the model to accurately simulate convection.

Using the Boussinesq approximation in a model to simulate the convection in black smoker vents finds that the temperature of the venting fluid should continue to rise as the temperature difference ΔT rises.[4] In measurements made of black smokers, however, the highest temperatures for fluid venting from them that has been observed is 405° C even though ΔT for these systems can be as high as $\sim$ 1200° C. These observations suggest that the Boussinesq approximation may not be valid under these conditions, indeed the density of water for temperatures over $\sim$ 200° C is a highly non-linear function of temperature differing from the linear form of the approximation.

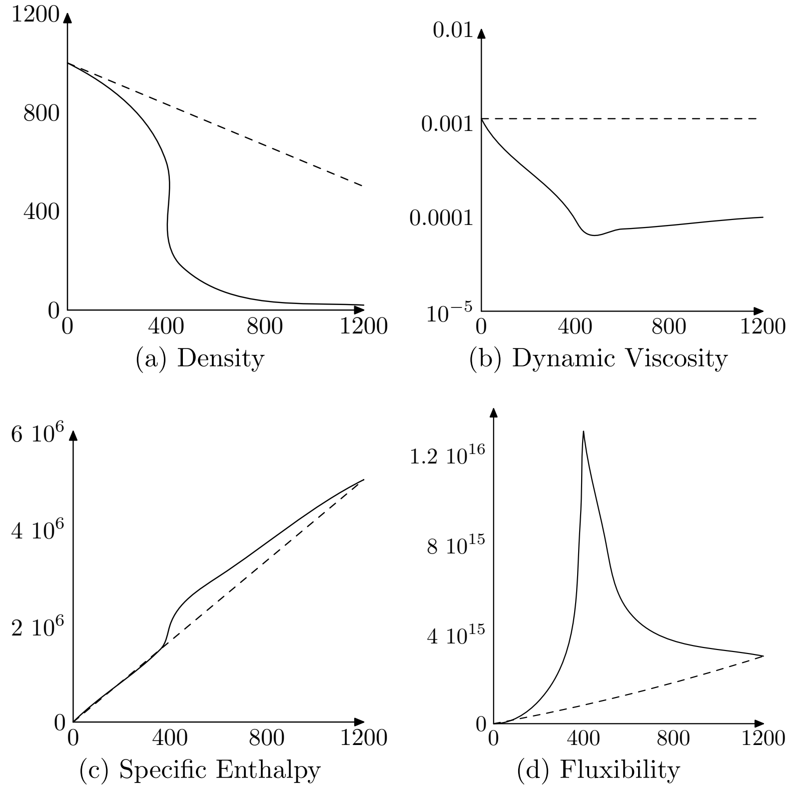

Figure 7: Graphs showing how thermodynamic properties of water change with temperature. As we can see the fluxibility, which is a measure of the ability of the fluid to transport thermal energy, reaches its maximum value at roughly 400° C. Figures adapted from [4].

The density of water is shown as a function of temperature in Figure 7a with the Boussinesq approximation plotted as a dashed line. As can be seen at temperatures above 200° C the Boussinesq approximation for the density differs considerably from the actual density.

The density of the fluid is also a function of the pressure of the fluid as well as its temperature, however for pressures above the critical pressure of 22 MPa the density curve in Figure 7a changes little compared with the change due to temperature.

As we can see in Figure 7a, for temperatures around $\sim 400^\circ\ C$ the density drops sharply. This shows that the Boussinesq approximation is not suitable for water at such temperatures, since this behavior is highly non-linear.

The next term in the expression for fluxibility is the dynamic viscosity μ. The variation with temperature of the value of this quantity for pure water is shown in Figure 7b, as well as the variation for a Boussinesq fluid, which is a fluid where the Boussinesq approximation applies. As we can see the values predicted by the Boussinesq approximation are far from the actual values, however the viscosity is only relevant in the boundary layer of the fluid meaning that for the broad features of the convection cell this is unimportant. We can also see that the viscosity of water is minimised at $\sim 400^\circ\ C$ meaning that the water has least resistance to motion at this temperature for the ranges found in black smoker systems.

The last quantity of interest to be able to determine how the fluxibility varies with temperature is the specific enthalpy. This is shown as a function of temperature in Figure 7c for both pure water and a Boussinesq fluid as before. The specific enthalpy of a fluid measures the amount of energy carried in the fluid as heat; the term $h_u - h_d$ measures this value relative to the down-flowing fluid at temperature T_d. As we can see from the figure the specific enthalpy of water does not differ significantly from the Boussinesq approximation, which is

$$h_u - h_d \approx c_p(T_u - T_d). \tag{6.4}$$

Putting all of these terms together into the expression for the fluxibility (equation (6.3)) tells us how it will vary with temperature. A graph of fluxibility against temperature is shown in Figure 7d. As can be seen from this graph the fluxibility rises rapidly once the water reaches a gas like state, and reaches its maximum value at temperatures between $\sim 400^\circ\ C$ and $\sim 500^\circ\ C$. This means that at this temperature the water is best able to transport heat energy, meaning that the heat flux in the convection cell will be maximised when the water takes this temperature.

7 Conclusion

We have seen that the governing equations for the convection of fluid in a porous medium can be expressed as eq. (2.39) and (2.42). If these equations are then non-dimensionalised the parameters of the system can be condensed into a single variable called the Rayleigh number Ra, which can then be used to give the conditions under which convection may occur. Looking at the stability of the system we then saw that convection may only occur in systems where the Rayleigh number $Ra > 4\pi$. We have also seen that the thermal properties of water impose an upper limit on the temperature of the water in Black Smoker systems, since the water is best able to transport heat energy at temperatures around $400^\circ\ C$.

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