Hurricane Vortex Investigation

ECMM713: Modelling Applications and Case Studies

Candidate No. 048373

May 2010

1 Introduction

Hurricanes, or more generally tropical cyclones, are complex weather systems, the formation and dynamics of which are not fully understood. They are warm cored, low pressure weather systems which generally form over tropical oceans. Hurricanes can cause devastation if they reach land, causing extensive damage and loss of life if they strike in heavily populated areas. The destructiveness of a hurricane, or rather the cost in monetary terms of rebuilding after a hurricane has hit, is proportional to the cube of the velocity of the hurricane's wind.[1] The wind velocity of a hurricane is in turn linked to the temperature of the air making up the weather system. The threat of global warming has led to a desire by scientists to know how the wind velocities of hurricanes will change in a warmer climate.

While the full dynamics of a hurricane are governed by complex equations, many of the features of a hurricane can be approximated by making some basic assumptions.

In this case study I will look at a simplified model to determine the density and temperature profiles for an axisymmetric vortex given a prescribed velocity profile. I will be comparing the results I achieve to those produced by Smith in Ref. [4]. I will also look at the reverse problem of prescribing a temperature profile and calculating the density and velocity profiles. This will enable me to look at the possible effects of a warmer climate, and in particular the effects of rising sea temperatures, on the size of the wind velocity in the hurricanes produced by my model. I will then look at these results in comparison to the findings of Emanuel in Ref. [1] to see if my simple model of a hurricane shows any of the trends found in data of real storms.

2 Hurricane Vortex

To derive the governing equations for this approximation of the hurricane vortex I will use cylindrical polar coordinates (r, z) where r is the radial component and z is the vertical component. I will also assume an axisymmetric vortex since although it is not true that real hurricanes exhibit axisymmetry, this simplification will allow me to approximate the main features of a non-axisymmetric vortex without its added complexity. In this notation p is the pressure of the air, ρ is the density, T is the temperature, v is the wind velocity, g is the acceleration due to gravity and f is the Coriolis parameter.

In this simplified hurricane vortex the governing equations can be derived from the balance of the vertical and horizontal pressure gradients. In the vertical direction hydrostatic balance applies, which means that the vertical gradient of pressure balances the gravity term:

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g. \tag{2.1}$$

In the horizontal direction the pressure gradient balances the sum of the centrifugal $\left(\frac{v^2}{r}\right)$ and Coriolis (fv) terms, and it takes the form

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{v^2}{r} + fv. \tag{2.2}$$

In order to simplify the notation I am using I will define C as the sum of the Coriolis and centrifugal terms

$$C = \frac{v^2}{r} + fv. \tag{2.3}$$

There are then two equations for the pressure gradients in the hurricane, taking the form

$$\frac{\partial p}{\partial r} = \rho C \tag{2.4}$$

$$\frac{\partial p}{\partial z} = -\rho g. \tag{2.5}$$

Due to the fact that the pressure term appears in both of these equations it will be convenient to eliminate it at this point, leaving just two quantities of interest; the velocity and the density. This can be achieved by cross differentiation which gives two equations

$$\frac{\partial^2 p}{\partial z \partial r} = \frac{\partial}{\partial z}(\rho C) \tag{2.6}$$

$$\frac{\partial^2 p}{\partial r \partial z} = -g\frac{\partial \rho}{\partial r}. \tag{2.7}$$

Setting the pressure terms equal to each other in (2.6) and (2.7) then gives the Thermal Wind Balance equation: ;

$$\frac{\partial}{\partial z}(\rho C) = -g\frac{\partial \rho}{\partial r}. \tag{2.8}$$

This can be simplified if density is written in the following form

$$\rho = \rho_0(z) + \rho_a(r, z). \tag{2.9}$$

Here $\rho_0(z)$ is the reference or atmospheric density, which will be independent of r since it is not specific to the hurricane, and $\rho_a(r,z)$ is the density anomaly created by the hurricane. The density anomaly will have the property $\rho_a \ll \rho_0$ which means that the anelastic approximation will apply. The anelastic approximation is where the density anomaly can be neglected in the gradient wind balance term since it varies little with height when compared with the reference density. This means that

$$\frac{\partial \rho}{\partial z} \approx \frac{\partial \rho_0}{\partial z}. \tag{2.10}$$

Substituting this into equation (2.8) above then gives

$$\frac{\partial}{\partial z}(\rho_0 C) = -g\frac{\partial \rho_a}{\partial r} \tag{2.11}$$

$$\Rightarrow \frac{\partial \rho_a}{\partial r} = -\frac{1}{g}\frac{\partial}{\partial z}(\rho_0 C). \tag{2.12}$$

This leaves an equation relating the density (and therefore the temperature) in the simplified hurricane to the velocity of its wind. Given a velocity profile we want to then be able to solve this equation numerically for the density and temperature.

In order to solve equation (2.12) numerically the domain over which I wish to find a solution will first have to be discretised. This will involve defining a grid of the radial and vertical coordinates on which the numerical values are required. It will be useful to stay away from $r = 0$ since as the value of r approaches this point the centrifugal term $\frac{v^2}{r}$ will approach ∞. This will cause problems when trying to calculate these terms, so I will start the radial coordinate at $r = 1\ km$ to avoid this singularity. I will use a step size of Δr between the horizontal points and Δz between the vertical points of the grid, where z will be able to start from $z = 0$ since there is no division by height in equation (2.12).

Now that I have defined a grid over which the numerical solution is required, I will need a numerical expression for $\frac{\partial \rho_a}{\partial r}$ and $\frac{\partial}{\partial z}(\rho_0 C)$. I will use a first order finite differencing numerical approximation for these derivatives:

$$\frac{\partial \rho_a}{\partial r} \approx \frac{\rho_a^{i+1,k} - \rho_a^{i,k}}{\Delta r} + O(\Delta r) \tag{2.13}$$

where $O(\Delta r)$ represents terms of order Δr or smaller, which are small enough in magnitude that they may be neglected in the approximation. The indices of the ρ_a terms refer to the indices of the points in the radial and vertical directions respectively, so $\rho_a^{i+1,k}$ represents the value of ρ_a at the $(i+1)$th point in the horizontal direction and the kth point in the vertical direction on the numerical grid.

An analogous approximation will be used for the $\frac{\partial}{\partial z}(\rho_0 C)$ term. These together give the numerical expression

$$\frac{\rho_a^{i+1,k} - \rho_a^{i,k}}{\Delta r} = -\frac{1}{g}\frac{\rho_0^{k+1}C^{i,k+1} - \rho_0^{k}C^{i,k}}{\Delta z} \tag{2.14}$$

$$\Rightarrow \rho_a^{i,k} = \rho_a^{i+1,k} + \frac{g\Delta r}{\Delta z}\left[\rho_0^{k+1}C^{i,k+1} - \rho_0^{k}C^{i,k}\right]. \tag{2.15}$$

Here ρ_0 has only one index since it is a function of height only.

Since I now have a numerical expression for $\rho_a^{i,k}$ in terms of $\rho_a^{i+1,k}$, a boundary condition is required for the density on the outer boundary of our domain, where r is maximum, in order to be able to obtain a numerical solution.

For large values of r far from the centre of the hurricane the velocity of the wind will be zero. The boundary condition which I will impose on the density at points where $r = r_{max}$ is $\rho_a(r_{max}, z) = 0$ for all z in the domain. Here r_{max} is the maximum value of r in the numerical grid.

Once I have solved this equation to obtain a numerical approximation for the density anomaly of the system, the Boussinesq Approximation relating density and temperature can be used to give an approximation for the temperature anomaly as well. It makes sense to l look for the temperature to take the same form as the density;

$$T(r, z) = T_{ref}(z) + T_a(r, z) \tag{2.16}$$

where T_{ref} is the atmospheric reference temperature and T_a is the temperature anomaly. The Boussinesq approximation then gives the relationship between density and temperature as being[3, pg.118]

$$\frac{\rho_a}{\rho_0} \approx -\frac{T_a}{T_{ref}}. \tag{2.17}$$

In order to be able to solve these equations an expression for the form that the reference density and reference temperature take in a tropical atmosphere is required. A good approximation for the form of the reference density ρ_0 is

$$\rho_0 = \rho_s \exp\left(-\frac{z}{H_\rho}\right) \tag{2.18}$$

where ρ_s is the density at the surface of the sea, and H_ρ is the density scale height. This is the height at which the reference density will have decreased by a factor of e. An approximation for the reference temperature of the atmosphere is also required. The approximation I will use is

$$T_{ref} = T_{surf} - \Gamma z \tag{2.19}$$

where T_{surf} is the temperature of the sea surface and Γ is the adiabatic lapse rate which is the rate at which temperature dissipates with height. For dry atmospheres, which I will assume is the case for the purposes of this investigation, $\Gamma \approx 10\ K \cdot km^{-1}$.

I now have everything I need to be able to solve the governing equation (2.12) for a given velocity profile. I will start by looking at a simple velocity profile which is symmetric about the radius at which the maximum wind occurs

$$v(r, z) = \begin{cases} V_{max} \exp\left(-\frac{z}{H_v}\right) \sin\left(\frac{\pi r}{2R}\right) & \text{for } r < 2R \\ 0 & \text{otherwise} \end{cases} \tag{2.20}$$

where R is the radius at which the maximum wind velocity is achieved, called the radius of maximum wind, V_{max} is the maximum velocity of the wind which is achieved at R and H_v is the velocity scale height.

I will also be looking at the inverse problem: if the temperature and\or density profiles are prescribed then I wish to be able to solve the governing equation to obtain the velocity profile. This will allow me to investigate how an increase in temperature will increase the velocity of the wind. To derive a numerical expression for the velocity in terms of density I will again start with equation (2.12) and integrate both sides with respect to z. This then gives the expression

$$-g\int_H^z \frac{\partial\rho_a}{\partial r}dz = [\rho_0 C]_H^z = \rho_0(z)C(r,z) - \rho_0(H)C(r,H). \tag{2.21}$$

Here H is the maximum value for z in the numerical domain. This can then be rearranged to give a quadratic equation in v

$$v(r,z)^2 + frv(r,z) - \frac{r\rho_0(H)C(r,H)}{\rho_0(z)} + \frac{gr}{\rho_0(z)}\int_H^z \frac{\partial\rho_a}{\partial r}dz = 0. \tag{2.22}$$

Using the quadratic formula to solve for v then gives an equation which can be solved numerically to attain the wind velocity from the density (and in turn from the temperature)

$$v(r,z) = \frac{-fr \pm \sqrt{f^2r^2 + 4d(r,z) - 4\dfrac{\xi(r)}{\rho_0(z)}}}{2} \tag{2.23}$$

where

$$d(r,z) = \frac{gr}{\rho_0(z)}\int_H^z \frac{\partial\rho_a}{\partial r}dz \tag{2.24}$$

$$\xi(r) = r\rho_0(H)C(r,H). \tag{2.25}$$

Since this equation is being solved for the velocity $v(r,z)$ the value of $C(r,H)$ will be unknown. The velocity at this height will be relatively low, so the top boundary of the domain may be given the condition $\xi(r) = 0$. The velocity must take values which are real and nonnegative on order to have a physical meaning. This means that only one of the solutions to equation (2.23) will correspond to a solution which is physically possible. The solution which is of interest here is the one corresponding to taking the positive sign in the equation. This will give only nonnegative values for the velocity while taking the negative sign would give unphysical solutions.[3, pg.68]

To calculate the value of the velocity profile numerically I will need to find a numerical approximation for $d(r,z)$. As before a finite differencing method can be used to approximate the integrand, with a sum being used to approximate the integral. This leads to the approximation

$$\frac{gr}{\rho_0(z)}\int_H^z \frac{\partial\rho_a}{\partial r}dz \approx \frac{gr_i}{\rho_0(z_k)}\sum_{k=H}^{z_k} \frac{\rho_{a\ i+1,k} - \rho_{a\ i,k}}{\Delta r}(-\Delta z) \tag{2.26}$$

where steps of $-\Delta z$ are used since the integral runs from the top of domain downwards. To solve this equation a density or temperature profile will be required. I will be using a temperature profile of the form

$$T_a(r, z) = T_{max} \exp\left(-\frac{z}{H_T}\right) \exp\left(-\frac{r^2}{30R_{max}}\right) \tag{2.27}$$

where T_{max} is the maximum value of the temperature anomaly and H_T is the temperature scale height.

3 Results

Running the numerical simulations gives the following output:

Figure 1: The hurricane vortex simulation code calculating density and temperature from velocity

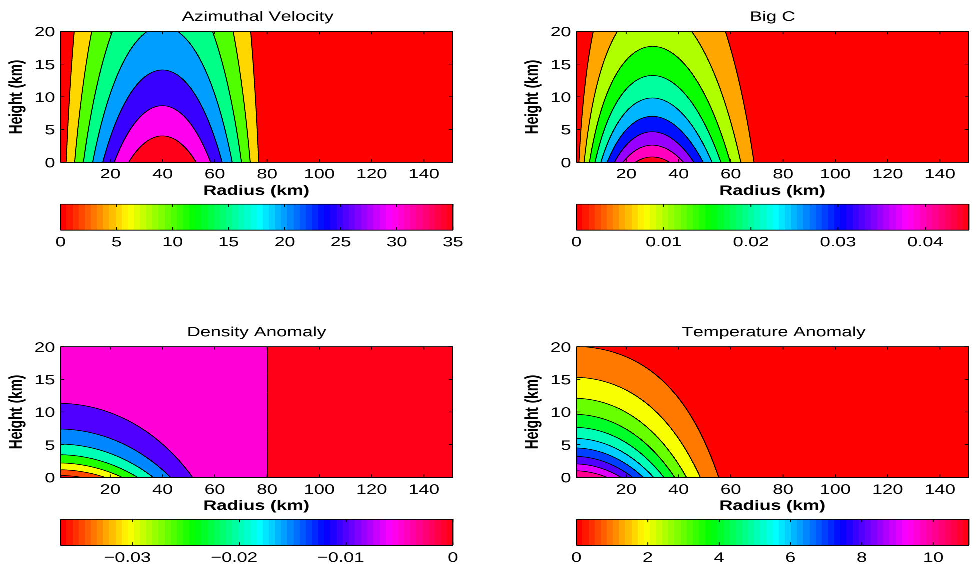


Figure 2: The reverse hurricane simulation code with temperature profile from forward code and the top boundary of the velocity profile specified as the original velocity to check if the code gives the same as the original input

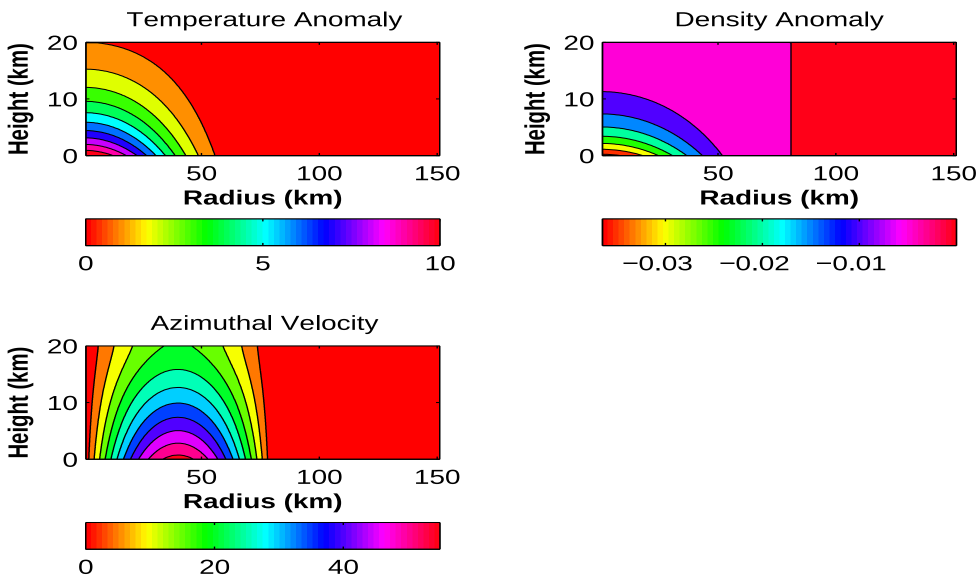


Figure 3: The reverse hurricane simulation code with temperature profile from forward code

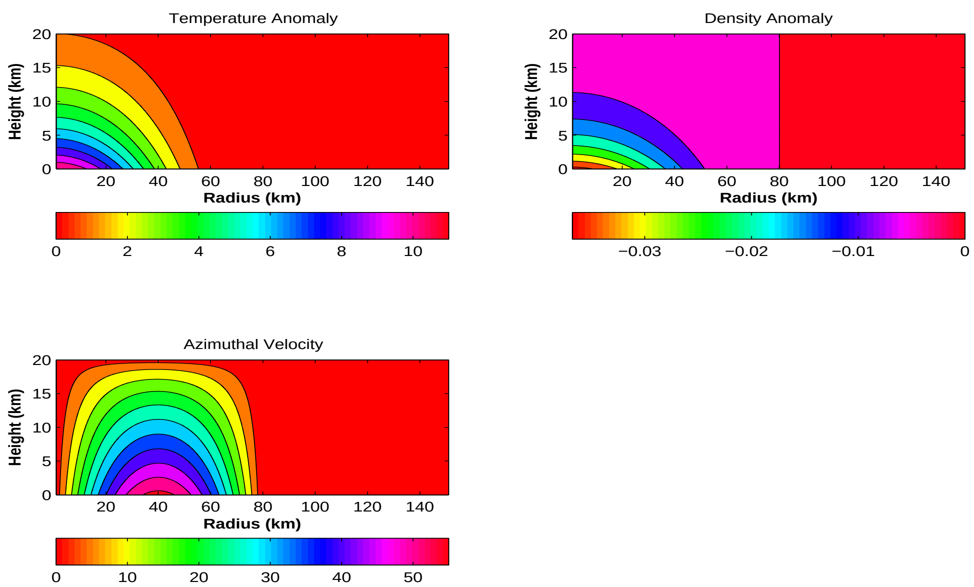


Figure 4: The reverse hurricane simulation run with a new temperature profile specified with $T_{max} = 10.5K$

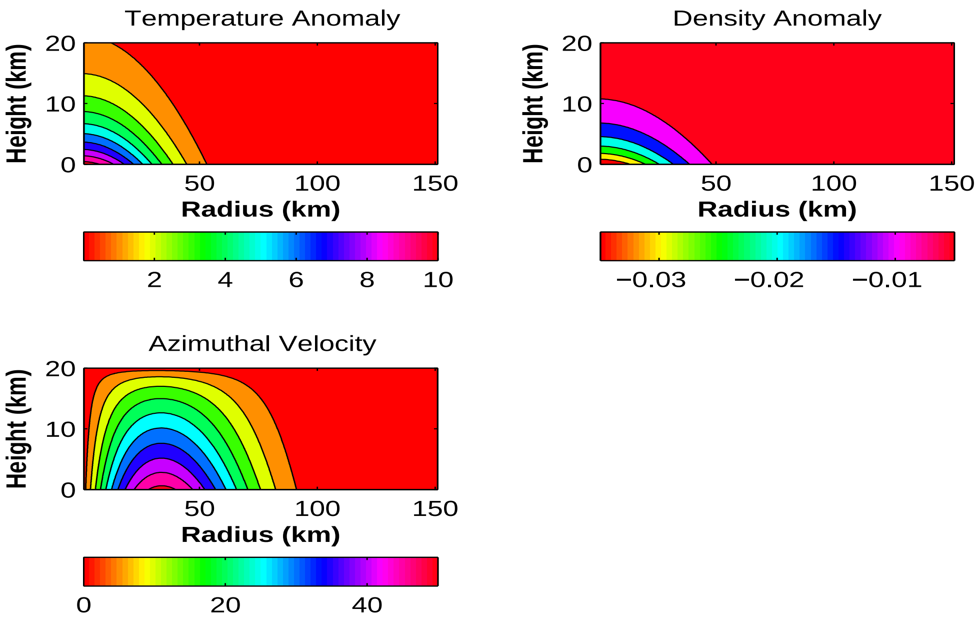


Figure 5: The reverse hurricane simulation run with a new temperature profile specified with $T_{max} = 11.5K$

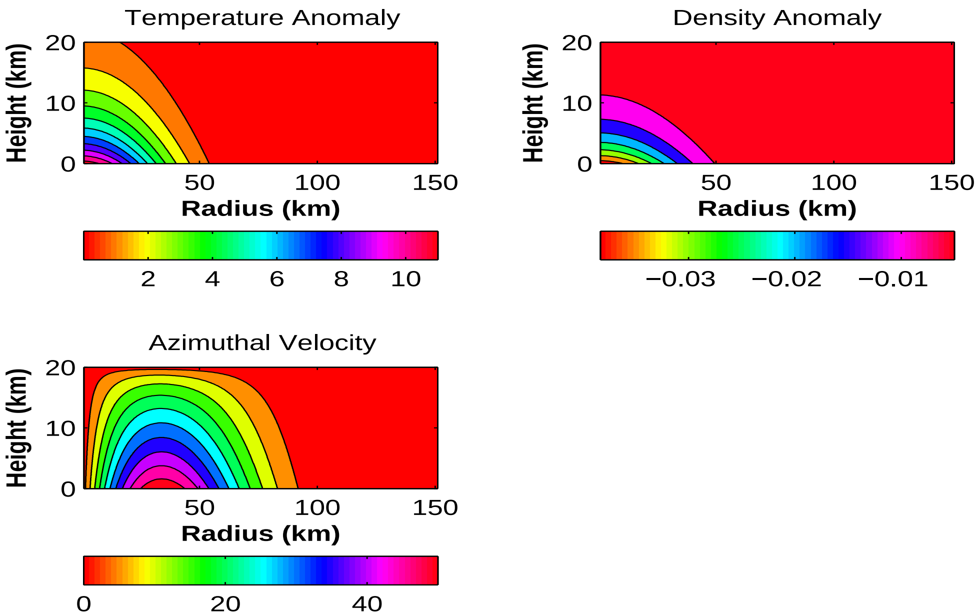
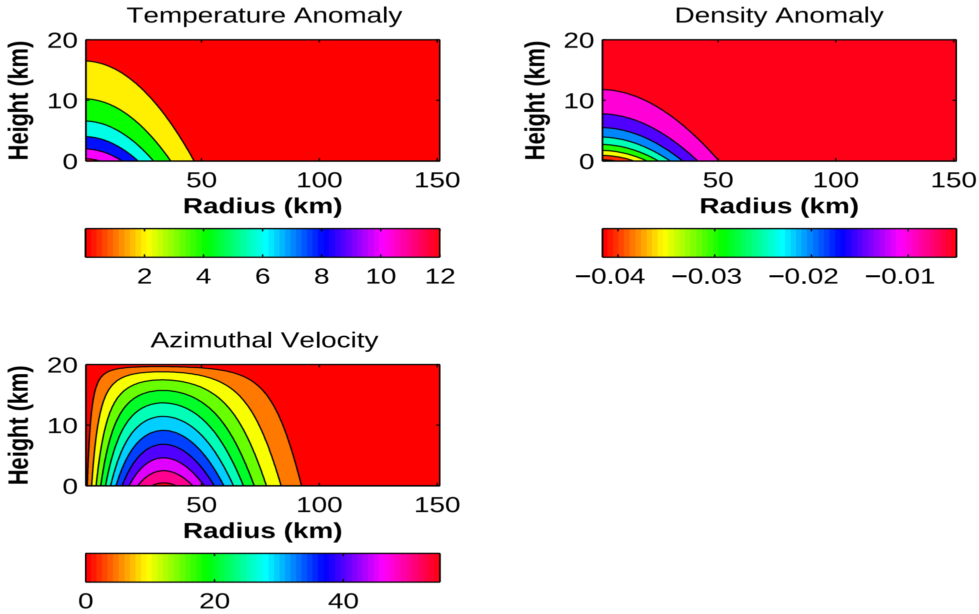


Figure 6: The reverse hurricane simulation run with a new temperature profile specified with $T_{max} = 12.5K$



4 Discussion

4.1 Accuracy of the Model

The model for the hurricane which I have used here is a simplified model, where some basic assumptions have been made in order to simplify the hurricane's dynamics while still capturing its main features. Here I will discuss the assumptions which have been made and how accurately they approximate the full dynamics.

One of the main assumptions made in this model is the assumption of an axisymmetric geometry in the hurricane vortex. This is not true for full hurricanes, in which the asymmetries play important roles in the intensification process along with their movement. The asymmetries in hurricanes arise from their interaction with the surrounding atmosphere and environment. One form of asymmetry which occurs in the fill dynamics of a hurricane is the "vortex Rossby wave" which can propagate angular momentum radially in the vortex. Research suggests that this process can play an important role in hurricane intensification.[5]

Numerical simulations[5] suggest that the process of hurricane intensification is inherently non-axisymmetric. In spite of these inherent asymmetries in hurricane vorticies their basic dynamics can be well approximated using an axisymmetric model. The intensification also does not pose a problem for the model used here since there is no time dependence included in either the governing equation (2.12) or the velocity (2.20) or temperature (2.27) profiles being used, so there will be no intensification being modelled.

To simplify calculations in the gradient wind balance, equation (2.4), the anelastic approximation has been used. The accuracy of this approximation is discussed by Smith in Ref. [4] where calculations using the anelastic approximation are compared with those completed using the full equation. He found that the maximum difference in the temperature anomaly between the exact method and the anelastic approximation takes a value of 1.35 K which occurs at the sea surface. The shapes of the two profiles obtained are very similar, leading to the conclusion that in an axisymmetric vortex the anelastic approximation gives an accurate calculation for the density and temperature anomalies, allowing the calculations to be simplified in this way without sacrificing a significant amount of accuracy.

In a full hurricane vortex the air moving in the boundary layer next to the sea surface will experience a frictional force which will cause it to slow down, meaning that the centrifugal and Coriolis terms will decrease in size while the horizontal pressure gradient will remain roughly the same. This will have the effect that this air will move inwards towards the centre of the vortex, where it will increase in velocity due to the principle of the conservation of angular momentum. This will affect the temperature anomaly in the storm since the air flowing close to the surface of the ocean causes water to evaporate, increasing its temperature. This in turn increases the pressure, and therefore density, gradients causing the storm to strengthen. In this model the gradient wind balance has been assumed meaning that this frictional force is not taken into account. This will effect the temperature at the core of the hurricane and should be kept in mind when looking at the resultant temperature and density profiles, although as with the case of axisymmetry the fact that there is no time dependence reduces the effect that this will have as its main effect is on hurricane intensification.

In the governing equation (2.8) the derivatives are approximated using a first order finite differencing method. For a grid with a step size of h this method has an error of order $O(h)$, so for the calculations made in this model the step size in the radial direction is $\Delta r = 1$ and the step size in the vertical direction is $\Delta z = 0.1$. This means that the error in using this method is of the order $O(\Delta r)$ since Δr is the largest of the step sizes. Reducing the step sizes in both the radial and vertical directions to $\Delta r = 0.1$ and $\Delta z = 0.01$ does not significantly change the size or shape of the density or temperature anomalies; the density anomaly changes by a factor of 0.0003 and the temperature by 0.08 while the running time for the code increases from roughly 3 seconds to nearly 1 minute. This is a large increase in running time for a relatively small increase in accuracy, so using step sizes of $\Delta r = 1$ and $\Delta z = 0.1$ seems reasonable.

Since the domain is already discretised the integral in the calculation of the velocity from density using equation (2.21) has simply been approximated as the sum of the value of the integrand at each of the grid points between the two endpoints of the integral multiplied by the step size Δz. This approximation will have an error of order $O(\Delta z)$ similar to the error in the approximation for the derivative. Reducing the step sizes by an factor of 10 again does not significantly change the output from the code for the velocity profile; the change is roughly 0.3 with a similar increase in computational intensity to that above. I was not able to reduce the step size further due to the size of the memory in the computer which I was using, however this accuracy seems to be sufficient for the purposes of this investigation.

4.2 The Forward Model

Prescribing the velocity profile discussed above in equation (2.20) and solving equation (2.15) using the Matlab code in Appendix A.1 gives the density and temperature anomaly profiles shown in Figure 1 alongside the velocity profile and that of C. The velocity profile prescribed corresponds approximately to a hurricane of moderate intensity with maximum tangential wind velocity $V_{max} = 40\ m \cdot s^{-1}$ occurring at a radius of $R_{max} = 40\ km$ from the centre and decreasing in magnitude with increasing height.

The density anomaly profile for this vortex is everywhere negative, with the minimum of 0.037 occurring in the centre at the sea surface. It then becomes less negative with increasing radius and height until it becomes zero at the top and outermost boundaries since there will be atmospheric density here. The centre of the hurricane having a lower density than that further out corresponds to the air being more buoyant than that surrounding it. This also corresponds to there being a pressure gradient through the relation (2.1).

The temperature anomaly within the hurricane has a negative radial gradient of the order of 10 K between the centre and the surrounding air at ambient atmospheric temperature on the boundary. The shape of the temperature anomaly profile is similar to that of the density anomaly which is what would be expected from using the Boussinesq approximation (equation (2.17)) to calculate it. The temperature at the core of the vortex is positive corresponding to a warm cored hurricane, with a negative temperature gradient in both the vertical and radial directions.

While the velocity profile I have used here is a simple approximation to the wind which would be experienced in a real hurricane, it provides a useful approximation to the profiles of the density and temperature anomalies which would be present. The approximation lacks some of the features of a real wind profile; indeed this approximation is symmetric about the radius of maximum wind whereas the velocity profile of a realistic hurricane is likely to obey a more asymmetric relation, with stronger winds towards the centre of the hurricane which will decay away more slowly with increasing radius such as detailed in Ref. [2]. Another feature of the velocity profile in a real hurricane which is missed by my approximation is that the radius of maximum wind would move radially outward with increasing height above the sea surface.

The profile for the temperature anomaly which I have obtained from the numerical calculation is somewhat different to that obtained by Smith in Ref. [4]. Smith's profile has a negative anomaly at the centre up to a height of 4 km with a warm core above this. The total variation in temperature anomaly over the whole of the domain is in the order of 10 K which is similar to the variation seen in Figure 1 from my simulation. In calculating his temperature profile Smith has calculated the pressure by integrating the gradient wind balance equation (2.4), and instead of using the Boussinesq approximation has related pressure and density to temperature through the ideal gas equation

$$p = \rho RT \tag{4.1}$$

where R is the specific gas constant for air. The velocity profile used by Smith in his calculations, while having the same maximum wind velocity and radius of my model, is not symmetric about this radius of maximum wind as mine is. The density anomaly is also more negative at the centre of the vortex which could explain why the temperature profiles differ. The reason for the difference in the structure of the temperature profile between mine and Smith's results is likely to be due to the difference in velocity profiles used between the two models rather than due to any of the other assumptions which I have made.

4.3 Testing The Forward and Reverse Models

To test the accuracy of the numerical method I have implemented in solving this simple model, I started by running the code in Appendix A.1 to calculate the density and temperature anomalies from the initial velocity profile (2.20), giving the density and temperature profiles plotted in Figure 1. I then used the temperature profile from this as the starting point for the code in Appendix A.2 and ran this code with the velocity at the top boundary specified as that of the initial velocity (2.20) at that point in order to see if I then obtained the original velocity profile back again. The output of this is displayed in Figure 2. As can be seen the density obtained from this temperature is the same as in Figure 1, however the velocity profile, whilst being a similar shape to the initial input is nearly twice its magnitude. In the velocity profile output by this code there were points which, although numerically equal to zero, were negative. This caused a lot of noise in the right hand portion of the velocity graph that is zero in the initial velocity input. It is likely that these points are simply artifacts of the numerical integration used and are of little importance. To present the graphs here I have set these points equal to zero so as not to distract attention from the parts of interest.

It is likely that the inconsistencies between the initial velocity and the resultant velocity are due to the numerical methods used for approximations to differentiation and integration. These approximations introduce errors at each step which on their own will be small enough that they do not significantly affect the result. However when they are used multiple times the errors which they introduce will begin to grow. The multiple use of numerical integration and differentiation, and therefore the growth in the errors, is probably what is causing the output from the reverse calculation to be twice as large as the original.

While ideally the velocity profile calculated here should be the same size as that originally specified, this code can still be used to give an idea of how the intensity of a hurricane will change in a warmer climate; it should simply be borne in mind that the velocities predicted using this code will be roughly twice as large as they should be. In changing the temperature of the climate we must then compare the relative velocities that this code predicts rather than their absolute magnitude.

4.4 Changes in Velocity Profile with Rising Sea Surface Temperature

As stated in the Introduction the monetary cost of the damage caused by hurricanes rises roughly with the cube of the velocity of the wind. Based on this, in Ref. [1] Emanuel defines an index for the destructiveness of tropical cyclones as

$$PDI = \int_0^\tau V_{max}^3 dt \tag{4.2}$$

where τ is the lifetime of the hurricane and V_{max} is the maximum velocity of the wind measured at a height of 10 m. He finds that the value of this index has increased markedly over the period since the 1970s and that the value of the index summed over an entire year is correlated with the average sea surface temperature for the corresponding year with $r^2 = 0.74$.

Although the index is integrated over the lifetime of a storm, I will use my simple model to look at the increase of sea surface temperature on the value of V_{max}^3 for the estimated wind speed for the steady state model.

The current average sea surface temperature in tropical regions is in the region of $27°C$[6] which corresponds to a temperature in Kelvin of roughly 300 K. The velocity profile predicted from my model for this value is shown in Figure 4. The maximum velocity of wind predicted for this value is $V_{max} = 51.47\ m \cdot s^{-1}$, giving a value of $V_{max}^3 = 1.36 \times 10^5\ m^3 \cdot s^{-3}$.

The model I am using has a limitation in that the relationship between density and temperature using the Boussinesq approximation (equation (2.17)) will not result in a lower density of air for a given increase in sea surface temperature, therefore the magnitude of the velocity will not increase. To simulate a rise in sea surface temperature for this model the value of T_{max}, and consequently the value of T_a, must be increased instead of the value of T_{surf}. Using values of $t_{max} = 10.5\ K$, 11.5 K and 12.5 K produces the velocity profiles shown in Figures 4, 5 and 6 respectively.

Theoretical predictions suggest that every increase in sea surface temperature of 1° should produce a corresponding increase in the maximum wind velocity of roughly 5%. Increasing T_{max} to 11.5 K, a 1° rise from its original value of 10.5 K, increases the wind speed to $V_{max} = 53.90\ m \cdot s^{-1}$, which is an increase of 4.7% from the original velocity. A similar increase to 12.5 K shows a similar increase of 4.3% suggesting that the model I am using is in line with other models in predicting the rise in wind velocity.

Increasing the maximum temperature anomaly by 0.5° to $T_{max} = 11\ K$ gives an increase in the maximum wind velocity to $V_{max} = 52.7\ m \cdot s^{-1}$, corresponding to $V_{max}^3 = 1.46 \times 10^5\ m^3 \cdot s^{-3}$. The increase in V_{max}^3 seen with this 0.5° temperature increase is slightly larger than 7%.

The increase in the mean sea surface temperature exhibited by Emanuel is roughly 0.5°, and the corresponding increase in PDI for the same period is 100%. The increase predicted by this model for the relevant escalation in temperature is far short of this. It does, however, agree with other theoretical predictions as to the increase that should be seen for an increase in sea surface temperature of this size. In a warming climate the lifetimes of storms will increase along with their maximum wind velocity. The model which I am using does not take into account this increased lifetime so will not account for the total rise in PDI, although even taking this into account Emanuel only estimates an increase of $8 - 12\%$ in the PDI.

This difference between predicted values for the increase in PDI and the actual increase seen in [1] suggests that in a warmer climate there may be other factors affecting hurricane wind velocity which are not take into account here such as the structure of the reference temperature in the troposphere.

5 Conclusion

The model which I have used here to model the dynamics of hurricanes, while not completely accurate, still seems to give a reasonable approximation for the relationship between the density and velocity profiles present in these storms. The estimated temperature and density profiles, while not having the same structure as those obtained by Smith in [4], do exhibit the same overall drop between the maximum value of the anomaly and the atmospheric values and the introduction of a more realistic wind profile would enable a more accurate analysis between the two sets of results to be made.

The model agrees with current theories as to the percentage increase in wind velocities which a rise in sea surface temperatures should bring. This is still short of the changes to maximum wind velocity which have been observed, and a possible extension to what I have done here would be to look at how changes to the structure of the reference temperature which a warmer climate may bring would affect the predicted maximum wind.

References

[1] Emanuel, K.A. 2005. 'Increasing destructiveness of tropical cyclones over the past 30 years'. *Nature.* **436**. 686-688.

[2] Holland, G.J. 1980. 'An Analytic Model for the Wind and Pressure Profiles in Hurricanes'. *Mon. Wea. Rev.* **108**. 1212-1218.

[3] Holton, James R. 1992. 'An Introduction to Dynamic Meteorology - 3rd Edition'. Academic Press Inc.

[4] Smith, R.K. 2006, 'Accurate determination of a balanced axisymmetric vortex in a compressible atmosphere'. *Tellus.* **58A**. 98-103.

[5] Smith, R.K. 2006. 'Hurricane Force'. *Physics World.* **19**. 32-37.

[6] http://research.metoffice.gov.uk/research/nwp/satelite/imagery/sst.html accessed 21st April 2010

A Appendix: Matlab Code

A.1 Code to calculate density and temperature given a velocity profile

```
%% Hurricane code to calculate temperature and density from given velocity
clear                    % clear the workspace

%% define constants
% set the step size
dr = 1 ;                 % delta r (km)  (step size)
dz = 0.1 ;               % delta z (km)  (step size)

r_max = 151 ;            % set the maximum value of r (km)
z_max = 20 ;             % set the maximum value of z (km)

R_max = 40 ;             % radius of maximum wind (km)
V_max = 40 ;             % maximum velocity (m s^-1)
H_v = 30 ;               % velocity scale height (km)
H_p = 9 ;                % density scale height (km)
rho_s = 1 ;              % surface density (kg m^-3)
g = 9.81    ;            % gravity (m s^-2)
f = 0.5e-4 ;             % coriolis term (s^-1)
T_surf = 300 ;           % sea surface temperature (K)
gamma = 10 ;             % adiabatic lapse rate (K km^-1)

I = (r_max-1)/dr + 1 ;   % calculate number of horizontal data points
                         % given range of r and step size since r
                         % starts at 1 not 0 we take 1 from r_max
K = z_max/dz + 1 ;       % calculate number of vertical data points
                         % given range of z and step size

%% define independent variables
r = 1:dr:r_max ;         % define vector r (km)
z = 0:dz:z_max ;         % define vector z (km)

%% initialise matrices of zeros
v = zeros(I,K) ;         % initialise velocity matrix
rho_a = zeros(I,K) ;     % initialise density anomaly matrix
C = zeros(I,K) ;         % initialise matrix for the variable C
T_a = zeros(I,K) ;       % initialise temperature anomaly matrix
T_ref = zeros(I,K) ;     % initialise reference temperature matrix

%% define velocity profile
% using given equation and also calculate C
for i = 1:I
    for k = 1:K
        if r(i) <= 2*R_max
            v(i,k) = V_max*exp(-z(k)/H_v)*sin((pi*r(i))/(2*R_max)) ;
                % no need to multiply r or z by 1000 since they're both
                % divided by values in km, so factors of 1000 would cancel
        else
            v(i,k) = 0 ;
        end
```

```
        C(i,k) = v(i,k)^2/(r(i)*1000) + f*v(i,k) ;
                        % need to multiply r by 1000 since it's in km
    end
end

%% calculate density
rho_0 = rho_s * exp(-z(:)/H_p) ;% calculate rho_0 from expression given

% calculate rho_a using the numerical appoximation we have derived
for i = I-1:-1:1
    for k = 1:K - 1
        rho_a(i,k) = rho_a(i+1,k) + (dr/(g*dz))*(rho_0(k+1)*C(i,k+1) -...
                     rho_0(k)*C(i,k)) ;
    end
end

%% calculate temperatire from density
% using boussinesq approximation
for k = 1:K
    % calculate reference temperature from expression given
    T_ref(:,k) = T_surf - gamma*z(k) ;
    for i = 1:I
        T_a(i,k) = - ( T_ref(i,k) * rho_a(i,k) )/rho_0(k) ;   % calculate
        % temperature from pressure using the Boussinesq approximation
    end
end

%% plot results
% azimuthal velocity
figure
subplot(2,2,1)
colormap(hsv) ;
pcolor(r,z,v') ;
contourf(r,z,v') ;
xlabel('Radius (km)','fontweight','bold')
ylabel('Height (km)','fontweight','bold')
title('Azimuthal Velocity')
colorbar('horiz')

% C
subplot(2,2,2)
colormap(hsv) ;
pcolor(r,z,C') ;
contourf(r,z,C') ;
xlabel('Radius (km)','fontweight','bold')
ylabel('Height (km)','fontweight','bold')
title('Big C')
colorbar('horiz')

% density anomaly
subplot(2,2,3)
colormap(hsv) ;
pcolor(r,z,rho_a') ;
contourf(r,z,rho_a');
xlabel('Radius (km)','fontweight','bold')
```

```
ylabel('Height (km)','fontweight','bold')
title('Density Anomaly')
colorbar('horiz')

% temperature anomaly
subplot(2,2,4)
colormap(hsv) ;
pcolor(r,z,T_a') ;
contourf(r,z,T_a') ;
xlabel('Radius (km)','fontweight','bold')
ylabel('Height (km)','fontweight','bold')
title('Temperature Anomaly')
colorbar('horiz')
```

A.2 Code to calculate velocity given a temperature or density profile

```
%% Reverse Hurricane to calculate velocity given temperature
clear                   % clear the workspace

%% define constants
% set the step size
dr = 1 ;                % delta r (km) (step size)
dz = 0.1 ;              % delta z (km) (step size)

r_max = 151 ;           % set the maximum value of r (km)
z_max = 20 ;            % set the maximum value of z (km)

R_max = 40 ;            % radius of maximum wind (km)
V_max = 40 ;            % maximum velocity (m s^-1)
T_max = 10.5 ;          % maximum temperature (K)
H_t = 9 ;               % temperature scale height (km)
H_v = 30 ;              % velocity scale height (km)
H_p = 9 ;               % density scale height (km)
rho_s = 1 ;             % surface density (kg m^-3)
g = 9.81     ;          % gravity (m s^-2)
f = 0.5e-4 ;            % coriolis term (s^-1)
T_surf = 300 ;          % sea surface temperature (K)
gamma = 10 ;            % adibatic lapse rate (K km^-1)

I = (r_max-1)/dr + 1 ;  % calculate number of horizontal data points
                        % given range of r and step size since r
                        % starts at 1 not 0 we take 1 from r_max
K = z_max/dz + 1 ;      % calculate number of vertical data points
                        % given range of z and step size

%% define independent variables
r = 1:dr:r_max ;        % define vector r (km)
z = 0:dz:z_max ;        % define vector z (km)

%% initialise matrices of zeros
v = zeros(I,K) ;        % initialise velocity matrix
rho_a = zeros(I,K) ;    % initialise density anomaly matrix
T_a = zeros(I,K) ;      % initialise temperature anomaly matrix
T_ref = zeros(I,K) ;    % initialise reference temperature matrix
```

```
d = zeros(I,K) ;          % initialise matrix for the variable d
const = zeros(1,I) ;      % initialise matrix for the variable const
v_temp = zeros(1,I) ;     % initialise matrix for the variable v_temp

%% define temperature profile
for k = 1:K
    % calculate reference temperature from expression given
    T_ref(:,k) = T_surf - gamma*z(k) ;

    % calculate temperature anomaly profile
    for i = 1:I
           T_a(i,k) = T_max*exp(-z(k)/H_t)*exp(- r(i)^2/(30*R_max)) ;
    end
end

%% calculate density
rho_0 = rho_s * exp(-z(:)/H_p) ;% calculate rho_0 from expression given

% calculate density anomaly using Boussinesq approximation
for i = 1:I
    for k = 1:K
        rho_a(i,k) = -( rho_0(k) * T_a(i,k) ) / T_ref(i,k) ;
    end
end

%% calculate velocity using the numerical approximation we have derived
% this part of the code is for defining the velocity on the top of the
% domain to check accuracy of forward then reverse code. The definition
% of the temperature profile needs commenting out to use this
%
% define const (xi) in terms of the velocity at the top of the domain
% use v_temp to temporarily hold the value of v at the top of the domain
% for i = 1:I
%     if r(i) <= 2*R_max
%         v_temp(i) = V_max*exp(-z(K)/H_v)*sin((pi*r(i))/(2*R_max)) ;
%     else
%         v_temp(i) = 0 ;
%     end
%     const(i) = rho_0(K) * (f * 1000 * r(i) * v_temp(i) + v_temp(i)^2) ;
% end

% work out d by integrating drho/dr
% the row d(K) will be zero, and each following row d(k) will be the sum
% from z=H to z(k), so to have zero velocity at the top of the domain I
% will just have to set const = 0 that way there's less chance of
% breaking the code if playing with the value of v at the top of the domain
for i = 1:I-1
    for k = K-1:-1:1
        % find numerical derivative of rho_a
        drhodr(i,k) = ( rho_a(i+1,k) - rho_a(i,k) ) / ( 1000 * dr ) ;
        % numerically integrate drhodr at row k and add it to the sum of
        % the integrals up to row k+1
        d(i,k) = d(i,k+1) - ( g * 1000 * r(i) / rho_0(k) ) * drhodr(i,k) * 1000 * dz ;
    end
end
```

```
% calculate v from quadratic formula
for i = 1:I
   for k = 1:K
        v(i,k) = - f * 1000 * r(i) / 2 + sqrt( (f^2) * (1000 * r(i))^2 - 4*d(i,k) +...
                     4 * const(i) / rho_0(k) ) /2 ;
   end
end

%% plot graphs
% temperature anomaly
figure
subplot(2,2,1)
colormap(hsv) ;
pcolor(r,z,T_a') ;
contourf(r,z,T_a') ;
xlabel('Radius (km)','fontweight','bold')
ylabel('Height (km)','fontweight','bold')
title('Temperature Anomaly')
colorbar('horiz')

% density anomaly
subplot(2,2,2)
colormap(hsv) ;
pcolor(r,z,rho_a') ;
contourf(r,z,rho_a') ;
xlabel('Radius (km)','fontweight','bold')
ylabel('Height (km)','fontweight','bold')
title('Density Anomaly')
colorbar('horiz')

% azimuthal velocity
subplot(2,2,3)
colormap(hsv) ;
pcolor(r,z,v') ;
contourf(r,z,v') ;
xlabel('Radius (km)','fontweight','bold')
ylabel('Height (km)','fontweight','bold')
title('Azimuthal Velocity')
colorbar('horiz')
```