Relativity in Periodic Geometry

ECMM705 - Mathematical Sciences Project 3

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1 Introduction

In 1905 Einstein published his famous paper "Zur Elektrodynamik bewegter Körper" ("On the Electrodynamics of Moving Bodies") in which he managed to resolve some of the big problems in physics at the time. One of these arose from the Michelson-Morley experiment, which was designed to determine the velocity of travel of the Earth through a hypothetical 'luminiferous aether' which was hypothersised to exist by Newton in order to explain some of the properties of light. This experiment had failed to detect any movement through this aether, casting doubt on its existence and suggested that the speed of light *in vacuo* was constant. Einstein showed this to be correct, and his Theory of Special Relativity changed the way that we think about space and time, unifying them as being similar quantities rather than separate entities as they were before.

The advent of Special Relativity, and later General Relativity, brought about new questions in physics. Special Relativity specifies the local properties of the new space-time quantity, but does not set any limitations on its global structure. This reopened a question which had been posed before but had not been solved. In the 19th century Nikolai Lobachevsky and Jnos Bolyai published papers on non-Euclidean geometry, leading to the question of whether space is Euclidean on a large scale. Gauss designed an experiment to try to answer this question by placing three observers on the tops of three different mountain and measuring the angle between the other two. The idea was that in non-Euclidean space the sum of the angles in a triangle is either more or less than 180°, so if space is non-Euclidean then the difference should be detected. The results of the experiment were not sufficient to be conclusive. Special Relativity brought this question to prominence and, although there have been many experiments designed to detect the topology of space-time (c.f.[4]), no-one has been able to find a definitive answer.

In this project we will look at Special Relativity and some of the unintuitive results it brings about, in particular time dilation and the resultant twin paradox. We will then look at the twin paradox in a periodic geometry and the complications which the geometry of space-time adds to the problem. Towards the end we will look at how hypothetical superluminal particles behave in terms of special relativity.

2 Special Relativity

In order that we are able to discuss Special Relativity we will require some definitions. Firstly we need to define what is called a *frame of reference*; a *frame of reference* is a coordinate system where an observer moving along with the frame can make measurements of distance and time. Special Relativity deals with special types of frames of reference called *inertial reference frames*, or simply *inertial frames*. An *inertial frame* is a frame of reference where Newton's first law, "a body continues in a state of rest or uniform motion unless acted upon by a force"[6, pg1] is satisfied; in particular an inertial frame does not undergo any acceleration. We can also define the *rest frame* of an observer as the inertial frame in which he is at rest.

Before Einstein developed special relativity the laws of mechanics were thought to obey the Galilean transformations. The Galilean transformations are transformations of coordinates between two different frames of reference S and S' moving relative to each other in the x direction with velocity V (see Figure 1). These transformations are:

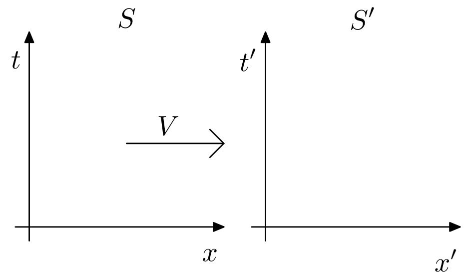

Figure 1: Two frames S and S' moving with relative velocity V

$$x' = x - Vt \tag{2.1}$$
$$y' = y$$
$$z' = z$$
$$t' = t \tag{2.2}$$

where (x, y, z) are the spatial coordinates in S and t is the corresponding time coordinate. The primed coordinates are just the corresponding elements in S'. These transformations together with the Galilean principle of relativity, "the laws of mechanics are the same in all inertial frames"[6, pg186], work fine for the relatively low velocities and distances which are familiar on earth. The Michelson-Morley experiment had however suggested that the velocity of light *in vacuo* (c) is a constant, which is contrary to the Galilean transformations. We can see this by first taking time derivatives of transformation (2.1)

$$\begin{aligned} \frac{d}{dt}(x') &= \frac{d}{dt}(x - Vt) \\ \Rightarrow \frac{dx'}{dt} &= \frac{dx}{dt} - V. \end{aligned} \tag{2.3}$$

We now have an equation relating the velocity measured by an observer in S' to the velocity measured by an observer in S. If we now take the velocity of light as measured by an observer in S to be c, we can then plug this into (2.3) to obtain the velocity measured by an observer in S'.

$$\frac{dx'}{dt} = c' = c - V \tag{2.4}$$

We can see that when $V > 0$ we have $c > c'$, therefore contradicting the result of the Michelson-Morley experiment. Also, Maxwell's equations for the propagation of electromagnetic waves are not invariant under the Galilean transformations. Whilst this

in itself was not axiomatically a problem (it simply resulted in rather messy equations), it was not ideal for some physicists and mathematicians who, like Einstein, felt that the equations governing the universe should be elegant, and should not be simple in one frame and messy in any other.
This lead Einstein to formulate his Special Theory of Relativity, which is based upon the following two axioms:[6, pg187]

- The laws of physics are the same for all inertial reference frames.
- The velocity of light *in vacuo* (c) is the same in all inertial frames.

These axioms obviously required a new set of transformations between reference frames since the Galilean ones disagree with both of these; Maxwell's equations are not invariant under Galilean transformation violating the first of the new axioms and, as we have seen, the Galilean transformations for velocity violate the second. This lead to the derivation of the Lorentz transformations between two inertial frames, which can be derived as follows. If we have two inertial frames S and S', with S' having velocity V relative to S, we can try a transformation between them of the form

$$x' = ax + bt. \tag{2.5}$$

The first of the new axioms tells us that we will then have the reverse transform taking a similar form

$$x = a'x' + b't'. \tag{2.6}$$

Here a, b, a' and b' are coefficients which we want to determine. At $x' = 0$ we have

$$0 = ax + bt$$

which we can then simplify to

$$-ax = bt$$
$$\Rightarrow -b = \frac{ax}{t} = aV.$$

We can do the same for the other equation, setting $x = 0$ and we obtain

$$-b' = \frac{a'x'}{t'} = -a'V$$

where the velocity is negative since S is moving with velocity $-V$ in the x direction relative to S'. Substituting b and b' back into (2.5) and (2.6) gives us

$$x' = a(x - Vt) \tag{2.7}$$
$$x = a'(x' + Vt'). \tag{2.8}$$

Now we want the transformations to be symmetric between S and S', again by Einsteins first axiom, so we must have $a = a'$. We will now invoke Einstein's second axiom, that the velocity of light is the same in all frames, therefore if

$$x = ct \tag{2.9}$$

then we must have

$$x' = ct'. \tag{2.10}$$

If we then send out a light signal at $t = t' = 0$ when $x = x' = 0$ it will have velocity c in both frames, therefore we can substitute (2.9) and (2.10) into (2.7) and (2.8), and using the fact that $a = a'$ we get

$$\begin{aligned} ct' &= a(ct - Vt) \\ ct &= a(ct' + Vt'). \end{aligned}$$

We can now multiply these together and solve for a, giving us

$$a = \frac{1}{\sqrt{1 - V^2/c^2}} \equiv \gamma. \tag{2.11}$$

Putting this back into (2.7) and (2.8) gives us

$$x' = \gamma(x - Vt) \tag{2.12}$$
$$x = \gamma(x' + Vt') \tag{2.13}$$

which are the direct and reverse Lorentz transformations for distance respectively. Since S and S' are moving only in the x direction we have trivial transformations in the y and z directions

$$\begin{aligned} y &= y' \\ z &= z'. \end{aligned}$$

We can also solve (2.12) and (2.13) for t and t' to give us the full set of Lorentz transformations;

$$x' = \gamma(x - Vt) \tag{2.14}$$
$$\begin{aligned} y' &= y \\ z' &= z \end{aligned}$$
$$t' = \gamma(t - Vx/c^2). \tag{2.15}$$

If the values of the velocity, V, and the distance, x, are small compared to c then we can see that $\gamma \simeq 1$ and equations (2.14) and (2.15) reduce to the Galilean transformations, so for the distances and velocities experienced on Earth the Galilean transformations suffice. Special relativity leads us to have to think about how we define things which before would have seemed rather trivial; one such example is the concept of length. If an object is at rest then it will be relatively trivial to define what its length is; it will be the distance from one endpoint to the other, and since the object is at rest it will not matter if the endpoints are measured at exactly the same time or not. When an object is moving, however, we must make sure that both endpoints are measured at the same time in the reference frame of the observer taking the measurements. We can see why by looking at an example of a moving rod. For simplicity we will assume that the rod's length is parallel to the direction of motion, and as before this will be parallel to the x axis. Now since the rod is moving, if we measure the first endpoint to be at the point x_1 at time

t_1 and measure the other endpoint to be at the point x_2 at a later time $t_2 > t_1$ then the first endpoint will have moved from the point x_1 where it was when it was measured, to the point $x_1 + V(t_2 - t_1)$. This will cause a discrepancy between the two possible measurements of length equal to $\pm V\Delta t$, where $\Delta t = t_2 - t_1$, and the sign depends on which endpoint was measured first. We therefore have to be careful to define the length measured for an object moving parallel to the x axis to be *the absolute value of the difference between the coordinates of its two endpoints, which must be measured at the same time*. We can see a concrete example of why this is especially important in special relativity in Section 2.9 but first we will look at some other concepts which are useful for looking at special relativity.

2.1 Lorentz Transformations for Velocities

Now that we have expressions for the Lorentz transformations for position it will be useful to derive the equivalent transformations for composition of velocities. We can see from the Lorentz transformations for position that

$$\begin{aligned} dx' &= \gamma\,(dx - V dt) &(2.16)\\ dy' &= dy \\ dz' &= dz \\ dt' &= \gamma\left(dt - V dx/c^2\right). &(2.17)\end{aligned}$$

We will now write the velocity in S as $\vec{u} = (u_x, u_y, u_z)$ where u_i is the component of velocity in the i direction where i represents each of x, y, and z in turn, and similarly for $\vec{u'} = \left(u'_{x'}, u'_{y'}, u'_{z'}\right)$. Now

$$\begin{aligned} u'_{x'} &= \frac{dx'}{dt'} \\ &= \frac{dx - V dt}{dt - V dx/c^2} \\ &= \frac{dt/dt - V}{1 - V(dx/dt)/c^2} \\ &= \frac{u_x - V}{1 - V u_x/c^2} &(2.18)\end{aligned}$$

where we have used the fact that $u_x = \frac{dx}{dt}$. We can do similarly for $u'_{y'}$ and $u'_{z'}$ which will then give us the Lorentz transformation for velocity

$$\begin{aligned} u'_{x'} &= \frac{u_x - V}{1 - V u_x/c^2} &(2.19)\\ u'_{y'} &= \frac{u_y}{\gamma\,(1 - V u_x/c^2)} &(2.20)\\ u'_{z'} &= \frac{u_z}{\gamma\,(1 - V u_x/c^2)}. &(2.21)\end{aligned}$$

2.2 Lorentz Transformations as Rotations and Rapidity

One consequence of the Lorentz transformations is the fact that the speed of light *in vacuo* acts as an upper speed limit for any massive object.

To see this we must first notice that the Lorentz Transformations can be thought of as being analogous to rotations of the coordinate axes. If we look at an event P in a frame S and the same event in a frame S' rotated relative to S at an angle θ (see Figure 2). The distance of event P from the origin will be r which will be the same in both S and S'. The angle that the line r makes with the x axis will be called ψ. We then have the relationships

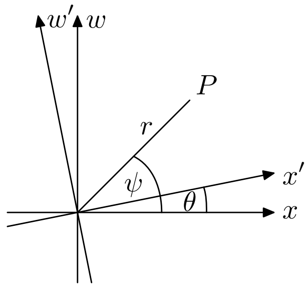

Figure 2: The coordinate axes S' rotated an angle of θ relative to S along with the point P which has an angle ψ relative to the x axis

$$x = r\cos\psi \tag{2.22}$$
$$w = r\sin\psi \tag{2.23}$$
$$x' = r\cos(\psi - \theta) \tag{2.24}$$
$$w' = r\sin(\psi - \theta)\,. \tag{2.25}$$

We can now expand the right hand side of x' and w' to give us

$$x' = r(\cos\psi\cos\theta + \sin\psi\sin\theta) \tag{2.26}$$
$$w' = r(\sin\psi\cos\theta - \cos\psi\sin\theta)\,. \tag{2.27}$$

Substituting in our expressions for x and w gives us

$$x' = x\cos\theta + w\sin\theta \tag{2.28}$$
$$w' = w\cos\theta - x\sin\theta. \tag{2.29}$$

If we now make a substitution for $w = ict$ and $w' = ict'$ we can see that we are left with

$$x' = x\cos\theta + ict\sin\theta \tag{2.30}$$
$$t' = t\cos\theta - \frac{x}{ic}\sin\theta \tag{2.31}$$

where in (2.31) we have simply divided through by ic. If we compare these to the Lorentz Transforms (2.14) and (2.15) we can see that they are equivalent if we have the relationships

$$\gamma = \cos\theta \tag{2.32}$$
$$\gamma V = -ic\sin\theta. \tag{2.33}$$

We can see that $\gamma = \cos\theta > 1$ so we must have the fact that the angle θ is imaginary. We can then write $\theta = i\phi$ which will turn (2.32) and (2.33) into

$$\gamma = \cos i\phi = \cosh\phi \tag{2.34}$$
$$\gamma V = -ic\sin i\phi = c\sinh\phi. \tag{2.35}$$

If we divide (2.35) by $c\cdot$(2.34) we then have

$$\frac{V}{c} = \tanh\phi \tag{2.36}$$

$$\Rightarrow \phi = \tanh^{-1}\frac{V}{c} \tag{2.37}$$

and we will define ϕ as the *rapidity*. Now we know that $-1 < \tanh\phi < 1$, which tells us $|V| < c$.

2.3 Four-Vectors

As we have seen in the previous section if we look at the quantity ict instead of t then we can interpret the Lorentz transformations as rotations of coordinate axes. This leads us to the idea of thinking of time as being a similar quantity to space, instead of a separate concept as it is in Euclidean space. We can then write time and space together as a four-vector

$$\vec{s} = (x_0, x_1, x_2, x_3) \equiv (ct, x, y, z) = (ct, \vec{x}). \tag{2.38}$$

This idea of space-time for $x, y, z, t \in \mathbb{R}$ is known as Minkowski four-space. We use ct instead of ict to avoid the added complication that the inclusion of imaginary numbers would have.
Now that we have the concept of a four-vector $\vec{s}$ for the position of a particle we must define the inner product of this vector with itself. This is usually defined as

$$\vec{s}\cdot\vec{s} = x_0^2 - x_1^2 - x_2^2 - x_3^2 \equiv x_0^2 - \vec{x}\cdot\vec{x} \tag{2.39}$$

where $\vec{x}\cdot\vec{x}$ is the usual scalar product for three-vectors. In general four-vectors are notated using capital letters in order to distinguish them from three-vectors and enabling them to be given the same letter as their three vector counterparts for simplicity. This means that the position vector for a particle is called $\vec{X}$.
We shall now define the proper time which will enable us to see derive the four-velocity of a particle.

2.4 Proper Time

Due to the fact that under the Lorentz transformations time t is not absolute between any two reference frames moving relative to each other it is useful here to define a quantity called the *proper time* of a particle τ. The proper time of a particle is defined as the time measured in the rest frame of that particle.
We will also need to define a distance quantity ds, which is defined as

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2. \tag{2.40}$$

This quantity is similar to the distance in $\mathbb{R}^3$ and is invariant under rotations and translations. It will be useful to define another invariant quantity which will be the ratio of ds^2 to c^2:

$$d\tau^2 = dt^2 - \frac{dx^2 + dy^2 + dz^2}{c^2}. \tag{2.41}$$

This quantity is called the proper time element and it is useful because it is the time which will be measured by an observer in their rest frame for the interval between two events occurring to that observer. It also enables us to look at the form of the four-velocity.

2.5 The Velocity Four-Vector

Now that we know the form of the position four-vector $\vec{X}$ and have the concept of proper time we can look at what form the velocity four-vector $\vec{U}$ takes. This velocity is defined as

$$\vec{U} = \frac{d\vec{X}}{d\tau} = \frac{d\vec{X}}{dt}\frac{dt}{d\tau}. \tag{2.42}$$

We can see from the definition of $d\tau^2$ that

$$d\tau^2 = dt^2\left[-\frac{1}{c^2}\frac{dx^2+dy^2+dz^2}{dt^2}\right] = dt^2\left[1-\frac{u^2}{c^2}\right] \tag{2.43}$$

$$\Rightarrow \frac{d\tau^2}{dt^2} = 1-\frac{u^2}{c^2} \tag{2.44}$$

$$\Rightarrow \frac{dt}{d\tau} = \frac{1}{\sqrt{1-u^2/c^2}} = \gamma \tag{2.45}$$

where $u^2 = \vec{u}\cdot\vec{u}$. This then enables us to simplify the expression for $\vec{U}$

$$\vec{U} = \frac{d\vec{X}}{dt}\frac{dt}{d\tau} = \gamma\frac{d}{dt}\left(ct,\vec{x}\right) \tag{2.46}$$

$$\vec{U} = \gamma\left(c,\vec{u}\right). \tag{2.47}$$

We now have an expression for the form of the four-velocity of a particle in terms of the familiar three-velocity. We will make use of this in Section 2.8 when we look at the four-vector forms of energy and momentum.

2.6 Causality

The fact that the speed of light *in vacuo* acts as an upper speed limit for massive objects allows us to tell if it is possible for any two events to be connected. Working with just one spatial dimension for simplicity, if the spatial distance Δx between two events is greater than their temporal separation multiplied by c, $c\Delta t$, then it is not possible for them to be connected, since a light signal will not be able to travel from one event to the other. If we define the interval between two events using the invariant distance (2.40) as

$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 \tag{2.48}$$

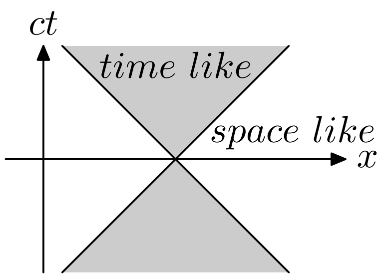

Figure 3: A light cone with the areas of space-like and time-like separation indicated.

then the value that Δs^2 takes will tell us if the two events can be related. If we have $\Delta s^2 > 0$ then it is possible for a signal travelling at less than c to travel from event 1 to event 2 and the two events are called time-like separated. It is possible for an observer to be at two separate time-like separated events. If $\Delta s^2 < 0$ then it will not be possible for a signal to travel between the two events and they are called space-like separated. These events cannot be related and it will not be possible for an observer to be at both of these events. The intermediate case is $\Delta s^2 = 0$ where the events can only be related by a signal travelling with velocity c. In this case the events are called light-like separated and the interval Δs^2 between them is called a null interval.

As an example we will take three events (t, x) $e_1 = (0, 2)$, $e_2 = (\frac{1}{c}, 2)$ and $e_3 = (\frac{1}{c}, 1)$. We can then see that

e_1 and e_2	$\Delta s^2 = 1 - 0 = 1$
e_2 and e_3	$\Delta s^2 = 0 - 1 = -1$
e_1 and e_3	$\Delta s^2 = 1 - 1 = 0.$

So we have events e_1 and e_2 are time-like separated, events e_2 and e_3 are space-like separated and events e_1 and e_3 are light-like separated.

It will now be useful to define the *world line* of a particle to be all of the points that particle visits in space-time. At any particular point on the world line of a particle we can draw the lines described by the null interval. The area enclosed by these lines is called the light cone for that point (see Figure 3). The light cone for a particular point contains all events which can be connected to it, so if the particle is a traveller the light cone encloses all the points that he is able to visit. In particular the world line of a particle must be contained within its light cone.

In analogy with the concept of space-like, time-like and null intervals we can define an arbitrary four vector as being space-like, time-like or null by looking at its inner product with itself. A four vector $\vec{A}$ will then be classified as space-like if the inner product $\vec{A} \cdot \vec{A} = A^2 < 0$, time-like if $A^2 > 0$ and null if $A^2 = 0$.

2.7 Metrics

The expression (2.40) is specific to a four-dimensional flat space-time. We can generalise this to n-dimensional curved space-times where it will take the form

$$ds^2 = \eta_{jk} dx_j dx_k \tag{2.49}$$

where the repeated indices j and k are summed from 1 up to n. The coefficients η_{jk} form an n by n matrix which is called the metric tensor. The metric tensor is used to specify the local geometry of space-time, which may depend on position, and is always symmetrical. For a flat 4-d space-time, which we are looking at here $\eta = diag(1, -1, -1, -1)$. We shall now look at relativistic energy and momentum.

2.8 Relativistic Energy and Momentum

The Newtonian momentum $\vec{p}$ of a particle has the form $\vec{p} = m\vec{u}$ where m is the mass of the particle. By analogy we can define the four-momentum $\vec{P}$ of a particle as

$$\vec{P} = m\vec{U} = (m\gamma c, m\gamma\vec{u})\,. \tag{2.50}$$

We now have the concept of relativistic mass arising, for if we define the relativistic mass of a particle as

$$m_{rel} = m\gamma \tag{2.51}$$

where m is the rest mass of the particle. We can then rewrite equation (2.50) as

$$\vec{P} = (m\gamma c, \vec{p_{rel}}) \tag{2.52}$$

where $\vec{p}_{rel} = m_{rel}\vec{u}$. Here we can see that as the speed $u \to 0$ $\vec{p}_{rel} \to \vec{p}$ so without ambiguity we can drop the subscript on the momentum three-vectors and set $\vec{p}_{rel} = \vec{p}$. To find an expression for the relativistic energy E of a particle we must now look at its relativistic mass m_{rel}. We can expand γ in powers of $\dfrac{u^2}{c^2}$ which, for particles having low velocities $u \ll c$, takes the form

$$m\gamma = m + \frac{1}{c^2}\left(\frac{1}{2}mu^2\right) + \cdots \tag{2.53}$$

where terms of higher order can be ignored as they will take values approximately zero. If we multiply this expression through by c^2 it then takes the form

$$m\gamma c^2 = mc^2 + \left(\frac{1}{2}mu^2\right) + \cdots \tag{2.54}$$

where the quadratic term $\left(\frac{1}{2}mu^2\right) = E_k$ is equal to the kinetic energy of the particle. If we now introduce a new quantity called the *rest mass* of a particle $E_{rest} = mc^2$ then the equation (2.54) can be written as

$$m\gamma c^2 = E_{rest} + E_k \tag{2.55}$$

and we can then interpret the left hand side of (2.55) as the total relativistic energy E of the particle

$$E = m\gamma c^2. \tag{2.56}$$

In different notation this equation becomes Einsteins famous equation $E = mc^2$. We shall make use of these quantities of relativistic mass and energy in later sections, but we will now look at some of the surprising consequences of using the Lorentz transformation. The first of these is length contraction.

2.9 Length Contraction

The Lorentz length contraction is an unexpected consequence of using the Lorentz transformations rather than the Galilean ones. It is the effect that a moving object will appear shorter than if it was at rest which can be demonstrated in the following way.

Take a space ship at rest in S' with its length parallel to the x-axis and its end points at x'_1 and x'_2, then its length in S' will be $L' = x'_2 - x'_1$ (see Figure 4).
For an observer in S, the spaceship will be moving with velocity V. We can then apply the Lorentz transformation to x'_1 and x'_2

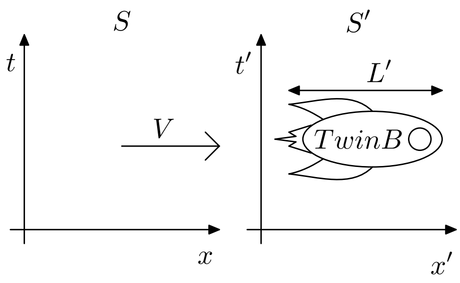

Figure 4: A spaceship moving with velocity V relative to S with proper length $L' = x'_2 - x'_1$

$$\begin{aligned} L' = x'_2 - x'_1 &= \gamma(x_2 - Vt_2) - \gamma(x_1 - Vt_1) \\ &= \gamma(x_2 - x_1 - V(t_2 - t_1)) \end{aligned}$$

Now both of these endpoints must be measured at the same time because of our definition of length, so $t_1 = t_2$ giving us

$$L' = x'_2 - x'_1 = \gamma(x_2 - x_1) > x_2 - x_1 = L. \quad (2.57)$$

Thus an observer in S measures the length of the spaceship to be less than he would if it was at rest, meaning that moving objects will appear shorter than if they were at rest relative to each other.
For example if a spaceship of proper length $100m$ is moving with velocity $V = 0.5c$ relative to the earth then an observer on the Earth will measure the length of the spaceship to be

$$L = \frac{L'}{\gamma} = \frac{100}{1.15} = 87m.$$

2.10 Simultaneity

Another surprising consequence of the Lorentz transformations is the loss of the absoluteness of simultaneity. Under the Galilean transformations, if two events are simultaneous in S then they are simultaneous in all other frames moving relative to S even if they occur at separate spatial locations x_1 and $x_2 \neq x_1$. This fits with the intuitive idea of simultaneity; if two events are simultaneous for one observer then we would expect them to be simultaneous for other observers. This is not the case however and we can see why this is in the following example.
We will take two events at points (t_0, x_1) and (t_0, x_2) in S which both have the same time t_0. If we now calculate the times when these occur as seen by an observer at rest in S' we obtain

$$t'_1 = \gamma(t_0 - Vx_1/c^2) \quad (2.58)$$

$$t'_2 = \gamma(t_0 - Vx_2/c^2). \quad (2.59)$$

We can now work out the time which the observer in S' measures between the events

$$t'_2 - t'_1 = \gamma(V(x_1 - x_2)/c^2). \quad (2.60)$$

This tells us that in S' the events do not appear to be simultaneous anymore unless $x_1 = x_2$. For values of V which are small compared to c we have $V/c \to 0$, so the events appear very close to simultaneous in S' which is what we should expect from everyday observations.

This relativity of simultaneity means that it is possible for an observer at rest in S to see an event e_1 occur before another event event e_2, while an observer in S' will see event e_1 occur after event e_2. For example if the observer in S measures the coordinates of event $e_1 = (0,0)$ and event $e_2 = \left(\frac{1}{2c}, 1\right)$ then the time he measures between these events is $t_2 - t_1 = \frac{1}{c} > 0$. If the relative velocity of the frames is $V = \frac{3}{4}c$ then the coordinates that the observer in S' measures for the events are $e'_1 = (0,0)$ and $e'_2 = \left(-\frac{1}{c}0.378, 0.945\right)$. If we now look at the time which this observer measures between the events $t'_2 - t'_1 = -\frac{1}{c}0.378 < 0$, so while the observer in S sees the first of the events to occur as event e_1, the observer in S' will see this as the second of the events.

2.11 Time Dilation

The effect of time dilation is one of the most famous consequences of special relativity and gives rise to the famous *twin paradox*. We can see its effect by looking at two events at times t'_1 and t'_2 at the point x' in S' with $t'_1 < t'_2$. An observer at rest in S' will measure the time between these events as $t'_2 - t'_1$. However an observer in S will measure the time interval between these events as $t_2 - t_1$, and we can find t_1 and t_2 in terms of t'_1 and t'_2 using the Lorentz transformation.

$$
\begin{aligned}
t_1 &= \gamma(t'_1 + Vx'/c^2) \\
t_2 &= \gamma(t'_2 + Vx'/c^2) \\
\Rightarrow t_2 - t_1 &= \gamma(t'_2 - t'_1 + Vx'/c^2 - Vx'/c^2) \\
\Delta t = t_2 - t_1 &= \gamma(t'_2 - t'_1) > t'_2 - t'_1 = \Delta t'
\end{aligned}
$$

This means that observers in S will see clocks moving relative to them running slowly compared to them. In particular, an observer in S will see an observer moving relative to him with velocity V take $\frac{1}{\sqrt{1-V^2/c^2}}$ seconds to count one second on his clock.

The effect of time dilation has been verified experimentally by looking at the decay of muons in the Earth's atmosphere. Muons are particles which decay shortly after creation into other elementary particles. A stationary muon has an average lifetime of roughly $2\times10^{-6}s$ and they travel at speeds close to that of light, c. If we take S to be the frame in which the Earth is relatively at rest and S' to be the frame in which the muon is at rest, we then have $\Delta t' = 2 \times 10^{-6}$ and we will take $V = 0.999c$. Without time dilation these particles would only be able to travel a distance of $0.999c \times 2 \times 10^{-6} \approx 600m$ before they decay. In reality muons are created in the upper atmosphere and many manage to reach the ground, travelling a distance of $\approx 15km$, much further than the $600m$ calculated. Taking into account the effects of time dilation manages to overcome this inconsistency.

Taking the values of $\Delta t'$ and V above then gives us that

$$\Delta t = \frac{1}{\sqrt{1-(0.999c)^2/c^2}} 2 \times 10^{-6} \quad (2.61)$$

$$= \frac{1}{0.0447} 2 \times 10^{-6} = 4.47 \times 10^{-5} s. \quad (2.62)$$

We can then work out the distance that the particles would be able to travel as seen by observers in S as $4.47 \times 10^{-5} \times 0.999c \approx 13.4km$ which is nearly enough to reach the ground where many are detected. The behaviors of muons then agree with the predictions of Special Relativity, and further experiments in particle accelerators have further confirmed the effects of time dilation.

Time dilation leads to interesting consequences, one of which is the *twin paradox* which we shall discuss in the next section.

3 The Twin Paradox

The twin paradox is a famous paradox in special relativity. It may be thought of as follows: two twins A & B start out together on the earth and twin B travels away in a spaceship before turning around and coming back, whilst twin A remains on the earth. The twins will have started out being the same age when twin B left, but when he returns he will be younger than twin A. This is because twin A will see B's time running slowly compared to himself due to the effect of time dilation which was discussed above. The paradox arises from the fact that B will also see A's time running slowly, since he sees A moving away from him and returning in the same way that A sees B leaving and returning. Since special relativity views all inertial reference frames as equal, twin B will expect A to be younger when they next meet.

We can express the time that twin A will expect twin B to measure for his outward journey as

$$t_1 = \gamma(t'_1 + Vx'_1/c^2). \quad (3.1)$$

Here t_1 is the time measured by twin A for the outward journey, t'_1 is the time measured by twin B for this journey, and x'_1 is the distance travelled by twin B as measured by himself for this portion of the trip. We can also similarly express the time that twin B will measure for his return journey according to twin A as

$$t_2 = \gamma(t'_2 + Vx'_2/c^2). \quad (3.2)$$

Now we can work out the total time that twin A will expect twin B to have measured for the length of the entire journey by adding (3.1) and (3.2) together and noticing the fact that $x'_2 = -x'_1$, since twin B's return route will be the reverse of his outwards route and hence the same distance, giving us

$$\begin{aligned} t_1 + t_2 &= \gamma(t_1' + Vx_1'/c^2) + \gamma(t_2' + Vx_2'/c^2) \\ &= \gamma(t_1' + t_2' + V/c^2(x_1' + x_2')) \\ &= \gamma(t_1' + t_2' + V/c^2(x_1' - x_1')) \\ &= \gamma(t_1' + t_2') \\ t_A &= \gamma\ t_B. \end{aligned} \tag{3.3}$$

Here t_A is the total time for the trip as measured by twin A and t_B is the total time for the trip as measured by twin B. Now we can see that

$$\gamma \equiv \frac{1}{\sqrt{1 - V^2/c^2}} \geq 1 \tag{3.4}$$

with = holding iff $V = 0$. Since twin B is moving relative to twin A we have $V > 0$ so we can now see that

$$t_A = \gamma\ t_B\ >\ t_B. \tag{3.5}$$

Twin A will then expect twin B to be younger when he returns to earth after his journey. We can rearrange this to express exactly how much younger twin A would expect twin B to be as

$$t_B = t_A\sqrt{1 - V^2/c^2}. \tag{3.6}$$

Now the paradox arises from the fact that we could also look at the situation from twin B's perspective and do similar calculations to in the case of twin A which would yield

$$t_B = \gamma\ t_A\ >\ t_A \tag{3.7}$$

with similar reasoning as before. This tells us that twin B would expect twin A to be the younger twin when they meet again, thus creating the apparent paradox.

The paradox is resolved by noticing the fact that for twin B to return to earth he must turn around, and in doing so he must accelerate. We can tell that it is actually twin B who changes his state of motion and not twin A by the fact that twin B has to exert a force in order to achieve his change in direction. In doing so he feels the effects of the acceleration, whereas twin A does not. This means that twin B is now no longer an inertial observer making his calculation invalid and resolving the paradox.[8]

In order to make twin A's calculations here correct the acceleration when twin B turns around must be infinite and instantaneous. The accelerations which occur when twin B leaves from, and returns to, the earth have also to be neglected in this argument. Although neither of these points will change the fact that twin A will be the older when the two meet again, they will mean that twin A's calculations will be incorrect in the amount that they predict for the age discrepancy of any "real life" situation where it is not possible for twin B to undergo infinite and instantaneous acceleration from one state of motion to another. In order to predict correctly the exact value for this discrepancy twin A will have to invoke general relativity when he makes his calculations (c.f. [7]), although that is beyond the scope of what I have set out to do here.

The prediction that twin A should be the older when the two eventually meet again has been experimentally verified using twin atomic clocks. Clock 'A' was left on the ground whilst clock'B' was taken on a commercial flight on a plane. The amount of time by which the moving clock and the sedentary clock differed was also consistent with the predictions of General Relativity.[9]

4 The Twin Paradox in Periodic Geometry

The example of the twin paradox which was discussed in the previous section supposes that the only way for twin B to return to his starting point is for him to have to turn around and return in the same way that he left. If, however, the topology of the space-time $\mathcal{M}$ which the twins inhabit is such that twin B is able to leave his starting point, travel away at a velocity V, and return to his starting point without undergoing any acceleration, it is more difficult to resolve the paradox. This is because both twins now travel with a constant uniform velocity along geodesics in $\mathcal{M}$. Therefore both twins appear now to be equivalent with no obvious difference between them; both twins see the other leave them, travel away and arrive back at their starting position without having undergone any acceleration. If both twins are in fact equivalent they would each be able to do similar calculations to those in the previous section which would both now appear to be valid, and we now have a paradox which is somewhat harder to refute than the previous form was.

4.1 The Topology of Space-Time

We will now consider a space-time $\mathcal{M}$ in which this property holds, i.e. one which is not simply-connected, and see how the twin paradox is resolved in this situation. A space-time which is simply-connected is one in which any path between two points may be continuously deformed into any other path between those points, or one in which all paths with the same endpoints are homotopic. This means that in a space-time which is not simply-connected there will be distinct paths between two points which are not homotopic to each other, i.e. where one path cannot be continuously deformed into the other.

As we saw in Section 2.3, in general an event in the space-time $\mathcal{M}$ can be expressed as a four vector (x_0, x_1, x_2, x_3) where x_0 is the time coordinate t and (x_1, x_2, x_3) are the spatial coordinates (x, y, z). We want our space-time to be locally identical to Minkowski space-time, which is $\mathbb{R}^4$ along with the flat Lorentzian metric

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

but we want it to have a different global structure. It is possible to obtain such a space-time by using a set of transformations called holonomies which will identify points in $\mathcal{M}$.[9] We will require that the time coordinate $x_0 = t$ is non-periodic since this would introduce closed time-like lines into $\mathcal{M}$ which would allow causality to break down in the sense that an effect would then be allowed to occur before it's cause. This would clearly be an undesirable property for our space-time to have. We can then specify that $x_0 \in \mathbb{R}$

and we will denote the three-dimensional spatial part Σ. Therefore we may write $\mathcal{M}$ as the topological product

$$\mathcal{M} = \mathbb{R} \times \Sigma \tag{4.1}$$

which will then satisfy this condition. Specifying the topology of our space-time $\mathcal{M}$ is then simply a case of specifying the topology of the spatial part Σ. We can express Σ in the form of a coset

$$\Sigma = X/\Gamma \tag{4.2}$$

where X is a simply connected manifold which is called the universal covering set and Γ is the holonomy group, since it is possible to show that any three-dimensional metric space can be expressed in this form.[9]

We then just have to choose the form of the transformations Γ. An obvious transformation Γ would be to identify the points (x, y, z) with the points $(x+\alpha, y+\beta, z+\lambda)$, so our space-time $\mathcal{M}$ would then take the form

$$(x_0, x_1, x_2, x_3) \equiv (x_0, x_1 + \alpha, x_2 + \beta, x_3 + \lambda). \tag{4.3}$$

We shall begin in the next section, however, by considering a space-time with only one spacial dimension, x, for simplicity, along with the usual time dimension, t. In a subsequent section we shall generalise our main result to a second spacial dimension from which generalisation to a given number of spacial dimensions will be relatively simple.

We will identify the points 0 and α in the x direction, then, so for any constant time t the spacial coordinates will form a circle (S^1) with circumference α. This means that the universe will have the form of an infinitely long cylinder ($\mathbb{R}^1 \times S^1$) with circumference α, where space runs around the cylinder and time runs along the infinite dimension (see Figure 5). The representation of the point (t, x) in space is now not unique, and we may write

$$(t, x) \equiv (t, x + n\alpha) \tag{4.4}$$

for any $n \in \mathbb{Z}$.

Although now the global properties of the space-time are different from those of the flat infinite Minkowski plane ($\mathbb{R}^2$) which was the topology under consideration in the previous section formulation of the twin paradox, the local properties do not change, meaning that special relativity remains locally valid in this new situation although as we shall see it will turn out not to be valid globally.

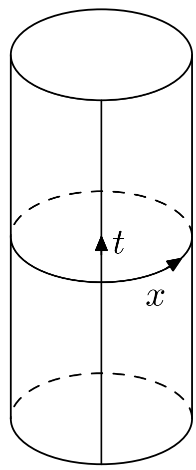

Figure 5: The cylinder ($S^1 \times \mathbb{R}^1$)

4.2 The Twin Paradox on the Cylinder

We may now reformulate the twin paradox in the following way. We can think again of two twins starting out together the same age on the earth at $(t, x) = (0, 0) = (t', x')$.

Again twin B leaves with velocity V, travels away, and returns back again to twin A. This time, however, twin B does not have to turn around in order to return to his starting point at $(0,0)$; he will be able to travel at a constant velocity V around the cylinder, starting at $x=0$, and returning to his starting point at twin A when he reaches $x=\alpha$ (see Figure 6). Twin A will then measure the time for the trip as $t_A=\alpha/V$. He will also calculate that twin B will measure the length of his trip as $t_B=t_A\sqrt{1-V^2/c^2}=\alpha/V\gamma$, thus concluding that twin B will be the younger when he returns.

For example, if the circumference of the universe is $\alpha=10c$ and on his journey twin B travels with velocity $V=0.75c$, then twin A will measure the time for the trip as $t_A=13.33s$, while he calculates that twin B will measure the time for the trip as $t_B=8.82s$. This means that twin A will expect twin B to be younger than him by $4.51s$. In this case, unlike in the original formulation of the twin paradox, the calculation which twin A makes for the difference in the ages of the twins will be exact if we do not look at accelerations at the start and end of the trip. In particular if twin B does not have to come to rest in twin A's reference frame then there is no need to invoke General Relativity in order to calculate the exact value of the age difference.

Twin B however may make similar calculations to twin A, as he did before, similarly concluding that twin A should be the younger after the trip by the same factor. This time there is no change of reference frame to invalidate twin B's conclusions, so we will have to find some other asymmetry between the twins journeys to account for the indiscrepancy.

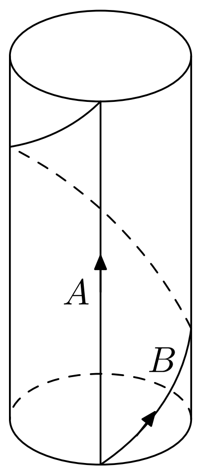

Figure 6: Twin B's journey around the cylinder whilst twin A remains at rest at $x=0$

Initially it seems that twin B should now be in an inertial reference frame, since he does not have to undergo any acceleration; the fact that he is travelling "around" the universe is merely an embedding of the 2-dimensional periodic universe in 3-dimensional space and does not correspond to any intrinsic curvature of the space itself. We could equally visualize the universe as a "flattened out" cylinder with infinite copies of A and B. However we will find that in this new universe there is a preferred rest frame, and that is the one occupied by twin A along with any and all observers at rest relative to him. This is distinct from the case of $\mathbb{R}^2$ where inertial frames are equivalent, with none being distinct from the others. This preferred rest frame on our cylinder is the only one in which an observer will measure two photons sent "around" the universe in opposite directions as returning back to him in equal times.[2] We can see this through some simple calculations. Take

$$x_B = Vt \tag{4.5}$$

$$x_{ph\pm} \pm n\alpha = \pm ct \tag{4.6}$$

where all measurements are taken in twin A's frame, x_B is the position of twin B and x_{ph} is the position of the photon. The sign refers to the direction of travel of the photon, whether it is travelling in the positive or negative direction, and also deciding the signs in the equation. The $n\alpha$ term comes from the identification (4.4) where the n can be interpreted as an index counting the number of times that the photon has traversed the universe. This index is an invariant independent of which observer is making the observations. We now equate (4.5) and (4.6) for the photon travelling in the positive direction

$$
\begin{aligned}
(4.6) \Rightarrow \quad & x_{ph+} = ct_+ - n\alpha & (4.7)\\
(4.5)\ \&\ (4.7) \Rightarrow \quad & Vt_{B+} = ct_{B+} - n\alpha \\
\Rightarrow \quad & n\alpha = ct_{B+} - Vt_{B+} \\
\Rightarrow \quad & t_{B+} = \frac{n\alpha}{c - V}. & (4.8)
\end{aligned}
$$

Similarly the photon travelling in the negative direction gives us

$$t_{B-} = \frac{n\alpha}{c + V}. \tag{4.9}$$

In both cases we can take $n = 1$ to correspond to B seeing each photon as having circled the universe once. Twin B will then have to conclude, from the fact that $t_+ > t_-$, that the circumference of the universe is longer in the direction he is travelling than in the opposite direction. If we now do the same calculations for twin A, instead of equation (4.5) we will find that

$$x_A = 0. \tag{4.10}$$

Following through the same calculations as before will now result in

$$t_{A+} = \frac{n\alpha}{c} \tag{4.11}$$

$$t_{A-} = \frac{n\alpha}{c}. \tag{4.12}$$

Again we may take $n = 1$ so that the photons will have travelled around the universe once each. Twin A will then conclude that the universe has the same circumference in both directions. Alternatively we may say that twin A occupies the only reference frame in which it is possible to observe photons circling the universe in opposite directions in equal times, since we may generalise twin B's calculations to any case where $V \neq 0$ and we will have the same result.

Following our example from before with $V = 0.75c$ and $\alpha = 10c$, twin B will find that the time taken for a photon to traverse the universe in the positive direction is $t_{B+} = 40s$ while it takes only $t_{B-} = 8s$ to travel in the negative direction. This means that he will measure the circumference of the universe to be $40c$ in the positive direction but only $8c$ in the negative direction. This raises the suspicion that there is something wrong with twin B's frame of reference on the cylinder.

Another way of seeing twin A's privileged position in the universe is to look at surfaces of simultaneity. We can define a surface of simultaneity as a hyper-surface of constant time, which is made up entirely of space-like separated events all of which occur at a particular time $t = t_0$. Twin A will be able to define consistent surfaces of simultaneity which will just be the circles S^1 at constant time $t = t_0$. However if twin B attempts to define surfaces of simultaneity he will run into problems at $0 = x = \alpha$ since at this point his surfaces will not match up with each other.

We can define x' and t' on the cylinder in terms of the Lorentz transformations

$$x' = \gamma(x - Vt) \tag{4.13}$$
$$t' = \gamma(t - Vx/c^2) \tag{4.14}$$

provided that we restrict x to the domain $x \in [0, \alpha]$, which will give us a domain for x' of[3]

$$x' \in [-Vt', \alpha/\gamma - Vt']. \tag{4.15}$$

We can now see that if we look at the x' coordinates at constant time $t' = t'_0$ then the points at $x = 0$ and $x = \alpha$, which correspond to the points $x' = -Vt'_0$ and $x' = \alpha/\gamma - Vt'_0$, clearly do not match up to each other on this surface for any $t_0 \in \mathbb{R}$. This is not the only inconsistency in B's surfaces of simultaneity and indeed there will be events on these surfaces which are time-like separated.[5] If we take for example the events $e'_1 = (0, 0)$ and $e'_2 = \left(0, \frac{3\alpha}{4\gamma}\right)$ on B's surface of simultaneity for $t' = 0$ then we can see that if we translate them into A's coordinates they will be

$$e_1 = (0, 0) \tag{4.16}$$
$$e_2 = \left(\frac{3V\alpha}{4c^2}, \frac{3\alpha}{4}\right). \tag{4.17}$$

Now due to the periodic nature of the space-time A can use the identification (4.4) to relabel e_2's coordinates as

$$e_2 = \left(\frac{3V\alpha}{4c^2}, -\frac{1\alpha}{4}\right). \tag{4.18}$$

Calculating the interval between these events we then find

$$\Delta s^2 = c^2\left(\frac{3V\alpha}{4c^2}\right) - \left(\frac{-\alpha}{4}\right) = \alpha^2\left(\frac{9V^2}{16c^2} - \frac{1}{16}\right). \tag{4.19}$$

Looking at this interval in the context of our example with $V = 0.75c$ and $\alpha = 10c$ then gives us

$$e_1 = (0, 0) \tag{4.20}$$
$$e_2 = \left(\frac{45}{8}, -\frac{10c}{4}\right) \tag{4.21}$$
$$\Delta s^2 = 2.54c > 0. \tag{4.22}$$

This tells us that these events are time-like separated as seen by A, and since the interval Δs^2 is invariant these events are time-like separated for B as well. Clearly this is not a desirable property for events on a surface of simultaneity to have. Indeed we would expect all points on a surface of simultaneity to be separated by space-like intervals if the surface is to be consistently defined, since by definition it is a surface of constant time where all events happen simultaneously. If we allowed time-like separated events to occur

on the same surface of simultaneity we would then run into problems with the concept of causality.
Twin B will also find that his space-time origin will repeat itself periodically.[2] To see this we will have to translate the identification (4.4) into (t', x') coordinates using the Lorentz transformations which will yield

$$(t', x') \equiv \left(t' - \frac{Vn\alpha\gamma}{c^2}, x' + n\alpha\gamma \right). \tag{4.23}$$

Now using (4.13) and (4.14) along with the identification (4.23) we can see that twin B's origin $(t', x') = (0, 0)$ repeats itself periodically at the coordinates

$$(t, x) = \left(-\frac{Vn\alpha\gamma^2}{c^2}, -\frac{V^2 n\alpha\gamma^2}{c^2} \right) \tag{4.24}$$

where $n \in \mathbb{Z}$ is any integer. It is clear, then, from these inconsistencies that there is something wrong with the use of Lorentz transformations in $\mathcal{M}$. We can see that the use of Lorentz transformations to go between the two sets of coordinate systems is not defined globally in this context; in $\mathbb{R}^2$ the transformations *are* well defined globally since there are no identifications of points. However in $\mathcal{M}$ they only make sense when used locally (since $\mathcal{M}$ is locally identical to $\mathbb{R}^2$), breaking down only when we try to use them on a global scale, that is for distances greater than α.
We are then forced to go back to the definition of proper time to calculate the times that A and B will measure for twin B's journey around the cylinder. Proper time $\Delta\tau$ for one spacial dimension, from (2.41), is defined as

$$(\Delta\tau)^2 = (\Delta t)^2 - \frac{(\Delta x)^2}{c^2} = (\Delta t')^2 - \frac{(\Delta x')^2}{c^2}. \tag{4.25}$$

For twin A $\Delta t_A = \alpha/V$ and $\Delta x_A = 0$ so

$$\Delta\tau_A = \Delta t_A = \frac{\alpha}{V} \tag{4.26}$$

which is what we would expect. For twin B, $\Delta t_B = \alpha/V$ again, but $\Delta x_B = \alpha$ meaning that

$$\Delta\tau_B = \sqrt{\frac{\alpha^2}{V^2} - \frac{\alpha^2}{c^2}} = \frac{\alpha}{V\gamma} \tag{4.27}$$

which is what twin A would predict using the Lorentz Transforms according to our earlier calculations. If we use (t', x') coordinates instead of (t, x) coordinates to do the same calculations we achieve the same results, since $\Delta t' = \alpha V\gamma/c^2$, $\Delta x'_B = 0$ and $\Delta x'_A = -\alpha\gamma$. This is exactly what twin A will predict each of the clocks to read at the end of the trip. Picking up from our example above, we have $\Delta\tau_A = 13.33s$ which is the same as t_A by definition, and $\Delta\tau_B = 8.82s$, the same as the time t_B which twin A predicted he should measure using the Lorentz transforms.
We can see, by using the definition of proper time to work out the times which the twins' clocks will show for the journey we recover the fact that twin A will always measure the

larger proper time. In fact we can see that the longer the spatial distance covered by an observer in a fixed time Δt, the shorter the proper time $\Delta\tau$ they will measure. We can also conclude that twin A may still use the Lorentz transformations to translate between his and twin B's coordinates since his calculations are consistent with the definition of proper time as long as he restricts his calculations to the domain $x \in [0, \alpha]$.

4.2.1 Coordinate Patches on the Cylinder

We have seen that the Lorentz transformations do not work globally on the cylinder. If we now set up coordinate patches on the cylinder we can solve the problem of changing between twin A's and twin B's coordinates. In each coordinate patch special relativity will hold over the whole of the patch, so we will be able to translate between (t, x) and (t', x'), however in at least one of the regions where the patches overlap, the values of (t', x') will not match up with each other. A coordinate patch is an open set which gives a unique coordinate representation for all events contained within it. The union of the coordinate patches also covers the whole of $\mathcal{M}$. On a cylinder at least 2 coordinate patches are required, for example we could have patch I running from 340° to 200° which would map to the open set $(-20, 200)$ on the x coordinates and patch II running from 160° to 20° which would map to the open set $(160, 380)$. We then have two regions where the patches overlap which we will denote μ and ν, where μ is the region between 160° and 200° and ν is the region between 340° and 20° (see Figure 7). In order that the two patches form a coordinate system on the cylinder we must have *transition functions* between the two patches in the overlap regions μ and ν. In the region μ the transition function will just be the identity function, and in the region ν we will need to add 360 to the x coordinate for patch I to obtain the x coordinate for patch II. In both cases the transition function for the time coordinate is trivial since the patches will run up the cylinder parallel to the t axis.

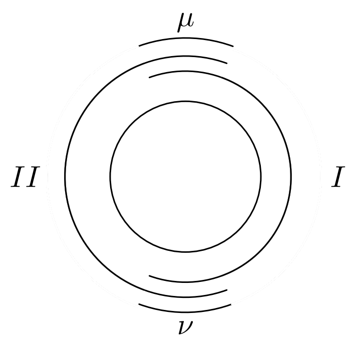

Figure 7: The coordinate patches on a constant time hyper-surface of the cylinder.

If we apply these patches on our cylinder, patch I will be the open set $\left(\frac{-1}{18}\alpha, \frac{5}{9}\alpha\right)$ on the x coordinates and patch II will be the set $\left(\frac{4}{9}\alpha, \frac{19}{18}\alpha\right)$. The patches will encompass the whole of the t coordinates. In this case, for the overlap region ν the transition function will be

$$(t, x + \alpha)_I = (t, x)_{II}, \tag{4.28}$$

where the subscript denotes the patch which the coordinates correspond to, while for the region μ it will still be the identity. We can now work out what the transition functions will be for twin B's (t', x') coordinates. In the region μ we will have $(t, x)_I = (t, x)_{II}$, so we will have the trivial transition function

$$(t', x')_{I'} = (t', x')_{II'} \tag{4.29}$$

for the region μ. To work out what the transition function for the region ν will be we first use the fact that in patch I we have

$$(t', x')_{I'} = \left(\gamma\left(t - \frac{Vx}{c^2}\right), \gamma\,(x - Vt)\right)_I . \tag{4.30}$$

We can then use (4.28), which will give us

$$(t', x')_{II'} = \left(\gamma\left(t - \frac{V(x+\alpha)}{c^2}\right), \gamma\,((x+\alpha) - Vt)\right)_I . \tag{4.31}$$

Putting (4.30) and (4.31) together then gives us the transition function for (t', x') coordinates in ν as

$$\left(t' - \frac{\gamma V\alpha}{c^2}, x' + \gamma\alpha\right)_{I'} = (t', x')_{II'} . \tag{4.32}$$

From this we can see that only twin A will have transition functions which are trivial w.r.t. time. It is now clear why we were able to use the Lorentz Transformations when we were working in $\mathbb{R}^2$ but not on the cylinder, since $\mathbb{R}^2$ requires only one coordinate patch to cover it while the cylinder requires at least two.
We will see an concrete example of how the coordinate patches work in Section 5.2.

4.3 The Twin Paradox on the Torus

If we now look at a space-time with two spatial dimensions instead of just the one which we have looked at so far, we can generalise the analysis to multiple spacial dimensions. As before we must choose the spatial part Σ of our space-time, which again is equivalent to making a choice for Γ. In the case of two spacial dimensions the choice of Γ can yield any one of five different spacial sections without introducing curvature into Σ; the Euclidean plane ($\mathbb{R}^2$), the Cylinder, the Möbius Strip, the Klein Bottle and the flat 2-Torus (T^2).[9]
We will look at $\mathcal{M} = \mathrm{T}^2$ since it is the simplest of these with both of its spacial dimensions periodic. The space-time is then a continuation of the identification (4.3) which we used earlier but with two spatial dimensions instead of three.
This time we will look at more than two 'twins', since there is now more than one possible direction in which the travelling twin may depart. Strictly speaking if we have more than two 'twins' they should not be be called twins, if there are three they should be called triplets, but we shall continue calling them twins to preserve the language used earlier, it will also enable us to still call the problem the *Twin Paradox*. We will again call the twin who remains in the absolute rest frame twin A, but this time there are two directions in which the travelling twin may travel in order to circle the torus once and return to his

starting point. We will therefore call the travelling twins B and C, and they will travel around the Torus in perpendicular directions; twin B will be travelling "around" the hole while twin C will be travelling "through" the hole (see Figure 8).

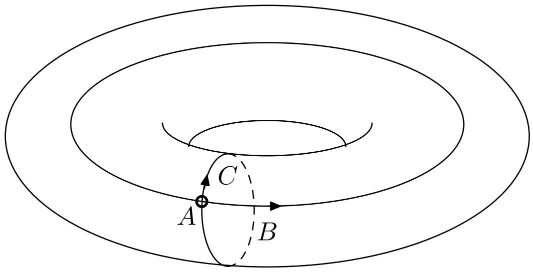

Figure 8: The Torus T^2 with twins B and C's world lines superimposed onto it.

On T^2 the paths of the travelling twins B and C will look like closed loops if we project their world lines onto a constant time hyper-surface of $\mathcal{M}$. We can denote these curves $\delta_i(u)$ for the parameter $u \in [0,1]$ where $\delta_i(0) = \delta_i(1)$ is the start-/end-point of δ_i respectively. Two curves δ_i and δ_j are called homotopic ($\delta_i \sim \delta_j$) iff δ_i can be continuously distorted into δ_j, or more precisely if $\exists F(u,v) : [0,1] \times [0,1] \to X$ such that $F(u,v)$ is a continuous function and $\forall u \in [0,1],\ F(u,0) = \delta_i$ and $F(u,1) = \delta_j$. All loops which are homotopic to each other belong to the same homotopy class, denoted $[\delta_i]$, where $[\delta_i] = [\delta_j] \iff \delta_i \sim \delta_j$.

In the standard twin paradox the world line of the travelling twin will be homotopic to the world line of the sedentary twin, but as we have seen the former will not be an inertial observer, so his proper time will inevitable be smaller than that of the sedentary twin. On T^2, however, the homotopy class of twin A, $[\delta_A]$, is different from the homotopy classes of each of the travelling twins $[\delta_B]$ and $[\delta_C]$. We can then define the *winding number* as the number of times a loop goes around the universe in a specific direction. In the case of the cylinder the winding number is just the number of times a particular loop travels all the way around, and will be represented by a single integer n. For the Torus T^2 the winding number will be a pair of numbers (m,n) where m is the number of times the loop goes around the hole and n is the number of times the loop goes through the hole. Twin B's winding number will be $(1,0)$ and twin C's will be $(0,1)$.

The winding number of a loop is an invariant property of that loop and will not be affected by a change of coordinates and does not depend on the observer measuring it. This means that two observers will not be in the same homotopy class unless their world lines have the same winding number, so twins A, B and C will all have different homotopy classes. Of these twin A is the only observer having a winding number of zero, meaning that he will be the older when the three meet again. Of the twins with non-zero winding number it is not immediately obvious which should be the younger at this meeting. We can say for observers with winding numbers of the form $(m,0)$ that the one with the larger value for m will be the younger and analogously for observers with winding numbers of the form $(0,n)$. In general, however, we cannot say which of twin B and twin C will be the older by looking at their winding numbers alone, for instance if the distance around the hole is much larger than the distance through the hole then twin C could go through the hole many times while twin B only goes around once, and still be older than twin B when they meet again. In general then while we are able to compare the ages of observers with winding numbers of the same form (eg. $(0, n)$), we may not say anything about observers who have winding numbers of different forms without some other information.

If as well as the winding number we know the lengths of each spacial direction in the identification (4.3), then we are able to work out which twin will be the younger when they meet again at the ends of their journeys. To do this we can superimpose the world lines of all the observers onto the universal covering set X such as in Figure 9. We then know from the definition of proper time, which for two spacial dimensions now takes the form $(\Delta\tau)^2 = (\Delta t)^2 - (\Delta x)^2/c^2 - (\Delta y)^2/c^2$, that a shorter spacial length in X will correspond to a longer proper time. Therefore the twins will age in the order of the length of their spacial distances in X, with the one having the shortest distance ageing the most, which in this case would be twin A, through to the one with the longest distance ageing the least.

If for instance we have as before $\alpha = 10c$ and now $\beta = 4c$ then if twin A measures the time for the journeys of all travellers as being $t_A = 20$ then we see that the travellers in Figure 9 will measure proper times of $\tau_A = 20$, $\tau_B = 17.32$, $\tau_C = 19.60$ and $\tau_D = 16.85$. In fact C could travel through the hole 6 times and still return to A having aged more than B, recording a proper time of $\tau_C = 17.44$.

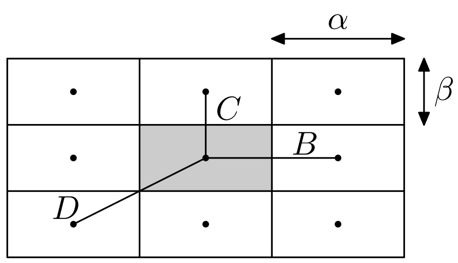

Figure 9: The universal covering set X with the world lines of B and C superimposed onto it. The world line D corresponds to an observer with winding index (-1, -1).

5 Superluminal Trajectories

While we have seen in Section 2.6 that the speed of light c acts as an upper speed limit for any massive object, there is nothing in Special Relativity which specifically forbids superluminal velocities. In this section we will look at the relativistic energy and momentum of particles travelling on superluminal trajectories and what restrictions this has on the mass of the particle. We will then look at a variation of the twin paradox which arises if we look at superluminal trajectories in a periodic space-time.

5.1 Relativistic Energy and Momentum for Superluminal Trajectories

As we saw in Section 2 the relativistic energy and momentum have the form

$$E = m\gamma c^2 \tag{5.1}$$

$$\vec{P} = (m\gamma c, m\gamma(u)\vec{u})\,. \tag{5.2}$$

If we consider a particle travelling with a superluminal velocity $V > c$, we can then see that the quantity

$$\gamma = \frac{1}{\sqrt{1 - V^2/c^2}} \tag{5.3}$$

cannot take real values since $V^2/c^2 > 1$. If we then look to the energy equation (5.1) we can see that the denominator will be complex, with real part equal to 0. Now for the quantity energy to make physical sense it must be real valued. This means that the numerator of the right hand side must be imaginary, but c is real valued meaning that the rest mass m of the particle must be imaginary.
For example if $V = 1.5c$ then

$$\gamma = \frac{1}{\sqrt{1 - (1.5c)^2/c^2}} = \frac{1}{\sqrt{1 - 2.25}} = \frac{1}{i\sqrt{1.25}} \approx \frac{0.894}{i}. \tag{5.4}$$

If the energy of the particle $E = 10c$ we then have its mass

$$m = \frac{E}{\gamma c} = \frac{10c}{0.894c} i = 11.180i. \tag{5.5}$$

If we increase the velocity of the particle to $V = 2c$ and keep the its mass the same we find that γ has now decreased to $\gamma = \frac{0.394}{i}$ meaning that the energy has also decreased, to $E = 6.455c$. This means that as the velocity of a particle decreases its energy increases, in particular $\lim_{V \to c^+} 1 - V^2/c^2 = 0$, so $\lim_{V \to c^+} E = \infty$ meaning that a particle travelling at superluminal velocities may not slow down and travel at subluminal velocities.
Looking now to the four-momentum (5.2) we can see that, as with the energy, it must be real valued since $(m\gamma) \in \mathbb{R}$, $c \in \mathbb{R}$ and $\vec{u}$ is real valued. If we look at the inner product of $\vec{P}$ with itself it will the the form

$$P^2 = (m\gamma)^2(c^2 - V^2) < 0 \tag{5.6}$$

since $V^2 > c^2$, so the four-momentum for superluminal particles is space-like as opposed to being time-like which is the case for subluminal particles.
Another curious property of particles capable of superluminal trajectories becomes apparent if we look at the definition of the proper time element and how this behaves for these particles. If we start with the definition (2.41) we find that it is helpful to write it in a different form in this case:

$$d\tau^2 = dt^2 - \frac{dx^2 + dy^2 + dz^2}{c^2} \tag{5.7}$$

$$= dt^2 \left(1 - \frac{1}{c^2}\frac{dx^2 + dy^2 + dz^2}{dt^2}\right) = \frac{dt^2}{c^2}\left(c^2 - V^2\right). \tag{5.8}$$

We know that for these particles $(c^2 - V^2) < 0$ and since $dt \in \mathbb{R}$, so $dt^2 > 0$, the expression for $d\tau^2$ is negative. When we take the square root of this to find a value for the proper time of a superluminal particles we find that, like their rest mass, this proper time must be imaginary.
If, for example, an observer is seen as travelling at a velocity $V = 1.5c$, and the time recorded by the rest observer is $t = 10$ then the proper time element that the observer will calculate is $d\tau^2 = -50$ meaning that the proper time is $d\tau = -7.07i$.

5.2 Superluminal Trajectories on the Cylinder

An interesting question arises from looking at the twin paradox in a periodic geometry. That is the following: "If one could travel on a superluminal trajectory, would they be able to return to where they started at an earlier time than they departed?", or alternatively "would the possibility of superluminal travel allow one to travel backwards in time?".
If we look at a space-time diagram for the world line of a superluminal particle on a non-periodic flat space-time such as Minkowski 4 space $\mathbb{R}^4$, we can see that it may be possible for a signal sent on a superluminal trajectory to travel backward in time. This may be achieved by first sending a signal on a trajectory forward directed in t, but backward in t', from starting point 1 to event 2, then sending a signal on a trajectory directed forward in t' but backward in t from event 2 to event 3 which will be at an earlier time than event 1 in both S and S' (see Figure 10). This does not pose a problem in the flat space-time in which it has been presented here because event 2 is space-like separated from event 1, so the observer in S may see 2 as happening after 1 while the observer in S' may see it occur before event 3 without problems. Here both the observers in S and the observers in S' see one of the trajectories as future directed and one as past directed. There will also be observers for whom both the trajectory from 1 to 2 and the trajectory from 2 to 3 are past directed, with 2 occurring between events 1 and 3, however there will not be any observers for whom both of these trajectories are future directed. This tells us that all observers will agree that for at least part of the trajectory of the signal is past directed, and they will all agree on which of event 1 and event 3 occurs first in time. The observers who will measure the shortest time between events 1 and 3 will be the ones who see event 2 occur exactly half way between 1 and 3. Indeed it must be the case that event 3 occurs before event 1 in this case since the signal must turn around at 2 thus allowing events 1 and 3 to be time-like separated, so if observers in S see 3 occur first then so must all inertial observers. The reverse situation would occur if there were observers who saw both halves of the trajectory to be forward directed, precluding the possibility of the signal arriving earlier than its departure even though there are some observers who see part of the trajectory directed into the past. This means that although superluminal travel would allow one to travel backward in time, it does not guarantee that this would be the outcome.
If however we look again to the cylinder with identification $(t, x) \equiv (t, x + \alpha)$ we can see that problems with causality may occur if we have the x coordinate of event 1 a distance of α away from the x coordinate of event 2. In the absolute rest frame S, if the x distance between the events 1 and 2 is equal to α, then they will occur at the same position in space on the cylinder. If then the superluminal "time traveller" observer departs from point 1 at time t_d and travels around the cylinder in a forward directed trajectory as seen

by the observers at rest in S, then these observers will all see the "time traveller" return to the point 2 at a time $t_r > t_d$, which will be later than she departed. The problem then comes if the trajectory which was forward directed in S is then past directed in S'. Initially it now seems that the observer in S' would see the return to be earlier than the departure, $t'_d > t'_r$, meaning that to the observers in one frame nothing extraordinary has happened, but in another the traveller has managed to travel backward in time along a past directed superluminal trajectory.

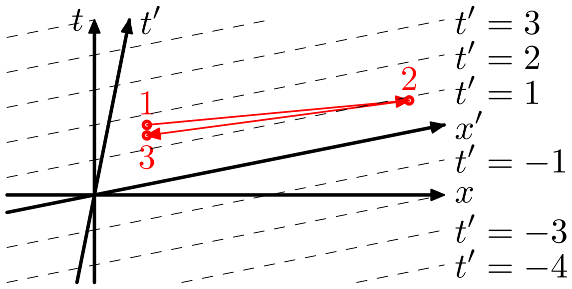

Figure 10: A possible journey by a superluminal particle travelling backward in time from event 1 to event 3 with the trajectory represented by the red arrows

In order to find out where the apparent contradiction arises we must first consider what we mean by *travelling backward in time*. If by *travelling backward in time* we mean arriving for a second time at a point marked on the cylinder, where the time of arrival is at an earlier time than the first visit to that point then there are sets of inertial observers who will say that the traveller has indeed travelled backward in time.[1] This definition however does not fit in with what most people mean when they think of time travel. They would consider *travelling backward in time* to be returning to an observer whom they had left at an earlier time than their departure, meaning that they could possibly stop themselves from leaving on the trip in the first place, the so called *time traveller paradox* (whilst the time traveller paradox is not the area of discussion here, it is useful to bring it up as it clearly demonstrates this other possible definition of *travelling backward in time*). As has been mentioned, if we take the first of these definitions of time travel we will indeed make it a possibility by allowing superluminal trajectories. However if we take the second definition the question of whether time travel is possible is not quite so easy to answer.

The answer will require us to look carefully at what we define to be the time measured for the journey of the "time traveller" by a given inertial observer. As we have seen above the proper time for superluminal objects is imaginary for all velocities $V > c$. If we want the time measured for the journey to have any physical significance, as it must if we are to use it to answer the question at hand, then we must exclude the proper time of the 'time traveller' as a possible definition. One possibility for the definition of the time measured in a frame S for the journey of a traveller would be the sum of local time intervals measured throughout the journey by observers at rest in S positioned at intervals around the cylinder. These local intervals may be less than zero if the trajectory happens to be past directed in the frame S, meaning that the sum of these will also be negative. Another possible definition of the total time for the journey would be the time measured on the clock of the observer whom the traveller leaves and rejoins at the end of her journey. This measurement will always be positive if the journey is forward directed in the absolute rest frame of the cylinder. The second definition tells us that if the trajectory of the traveller is forward directed according to an observer who is at rest in the absolute rest frame, and whom she will rejoin at the end of her journey, then there can be no observers who will claim that she returned before she departed to change events that occurred before she left. This result will be true even though for some of the observers on the cylinder the trajectory on which the traveller is moving is past directed, meaning their sum of time intervals measured throughout the journey is negative. In order to reconcile this difference we will have to go back to looking at the coordinate patches on the cylinder.

We will take the same coordinate patches as we had before on our cylinder; patch I being form 340° to 200° and patch II form 160° to 20°. As before we will have observer A at rest in the absolute rest frame, positioned at $x = 0$, and observer B at rest in the primed coordinate system, which is travelling at velocity $V = 0.75c$ relative to A. We will also imagine that in each of these frames we have other observers at rest to make suitable time measurements where required. We will then have the "time travelling" observer C travelling twice as fast as B, at $V = 1.5c$, according to A.

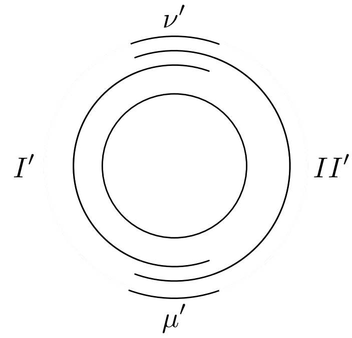

Figure 11: The positions of the primed coordinate patches at the instant when C reaches A for the first time. B will be directly opposite them on the cylinder.

Now when the traveller C returns to A's position for the first time along her journey, the primed observer B will have moved around the cylinder a distance of $\frac{1}{2}\alpha$ as seen by A, and the primed coordinate patches will have moved corresponding to the observers in the primed frame at rest relative to B having travelled around the cylinder, now being positioned as in Figure 11 with the observers making measurements in patch I′ being positioned where patch II′ was before, between 160° and 20°, and those making measurements in patch II′ positioned between 340° and 200°. If as before the circumference of the cylinder is $\alpha = 10c$ then observer A will measure a time of $t = 6.67\ s$ for this part of the journey. The time measured by B at this instant will be

$$\begin{aligned} t' &= \gamma\left(t - Vx/c^2\right) \\ &= \frac{1}{\sqrt{0.4375}}\left(\frac{10c}{1.5c} - \frac{7.5c}{2c}\right) \\ &= 4.41\ s. \end{aligned} \tag{5.9}$$

In order to measure the time of the reunion however the observer taking measurements should be the one who is now at the point $x = 0$. The time measured by this observer for this first reunion will be

$$\begin{aligned} t' &= \frac{1}{\sqrt{0.4375}}\left(\frac{10c}{1.5c} - \frac{7.5c}{c}\right) \\ &= -1.25\ s < 0. \end{aligned} \tag{5.10}$$

The observer measuring this time will not be twin B, however, it will be the observer who is at rest and positioned at 180° relative to B. This measurement will come about because in synchronising their clocks the observers in S' will use light signals, so the observer positioned at 90° relative to B will send photons towards both B and the observer at 180°. If C leaves B when the photon reaches him, C will reach the observer at 180° before the photon does, so this observer will measure a negative time for the journey.

So far the primed observers in S' have only seen the traveller C travel halfway around the cylinder, and she has remained in patch I$'$ for the entirety of the journey. This means that nothing has happened up until this point according to B which would not have happened in a non-periodic space-time such as $\mathbb{R}^4$. In order that we see the topology of the cylinder come into play we will have to have C circle the cylinder as seen by the primed observers. This will happen at the same instant as the event that B has circled the cylinder once according to A. This event will also be the first time when all three observers, A, B and C, are all reunited.

Shortly after the first time that C returns to twin A, she will leave patch I$'$ according to the observers in S' and so these observers will be forced to use patch II$'$ if they are to continue making measurements of C's voyage. Since this overlap region will be the region μ' there will not be any need to adjust the patch I$'$ measurements for them to correspond to those of patch II$'$ since the transition function in this section is simply the identity.

When the three observers A, B and C meet simultaneously for the first time we know from our discussion of the twin paradox in Section 4 that the proper time which twin B will measure for this event is $t_B = 8.82\ s$. This time corresponds to the time measured by observers in patch I$'$. The observers who are making measurements of C's journey are in patch II$'$ however so we can calculate the time which they will record for the reunion using the transition function (4.32) which gives us this time as

$$t'_{II'} = \left(t_B - \frac{\gamma V \alpha}{c^2}\right) \tag{5.11}$$

$$= -2.52\ s. \tag{5.12}$$

This time corresponds to the sum of local time intervals measured by the observers at rest in S', as they have seen C travelling at a velocity $V' = -6c$ relative to them, but does not correspond to the total time for the journey.

Although throughout the journey the observers at rest with respect to B will have seen the superluminal traveller C travelling consistently backwards in time, the final time for the journey must be positive as, if we disregard the presence of the time traveller C, the situation is identical to that discussed in Section 4 above. The presence of the traveller C here merely explicates the fact that we must reconcile the difference between the primed observers seeing C travelling backwards in time and the positive time difference between the departure of the superluminal traveller and the primed observers from the sedentary twin and their eventual reunion. The answer to why this apparent discontinuity occurs is that if we start by looking at a clock in patch I$'$ and using that as the starting time for the trip, we must then use the patch I$'$ time at the end of the trip as its finishing time. This positive time difference between the start and the end of the journey means that for the observers at rest in S' to claim that the traveller C is travelling backwards in time is equivalent to them comparing the clock in patch I$'$ at the start of the trip to the clock in patch II$'$ at the end of the trip and claiming that the total time for the trip is negative even though the observers clocks show positive time differences. If we want to use a clock in patch II$'$ to measure the final time for the journey then we must again use a clock in patch II$'$ to measure the time at the beginning of the journey, again giving a positive time for the trip. For the observers in S' to claim that C travels backwards in time would be for them to define time as the integral of local time differences, which would make this definition inconsistent with the definition of proper time as well as the intuitive idea of 'time'. The only observers for whom the sum of local time measurements is equal to the proper time for the trip are observers in the absolute rest frame; the privileged observers.

6 Conclusion

We have seen that the introduction of the Lorentz transformations to translate between the coordinates of moving inertial reference frames causes unintuitive phenomenon such as length contraction and time dilation to take place. We have also seen that in Special Relativity observers in inertial reference frames are in a special class of observers who may make use of the Lorentz transformations. The Twin Paradox explicates this fact since the travelling twin B is not in an inertial frame for the entirety of the trip, so must account for this when making calculations throughout the trip.
We have also seen that while in a flat Minkowski space-time $\mathbb{R}^4$ no observer may claim to be absolutely at rest, in a periodic space-time there is a privileged set of observers who occupy an absolute rest frame within the universe. These observers are able to make use of the Lorentz transformations on the whole of the space-time unlike other sets of observers who may only make use of them locally. In a periodic space-time it is important to pay attention to coordinate patches then making measurements of space and time in order to avoid apparent inconsistencies and paradoxes, especially when superluminal velocities are involved.

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